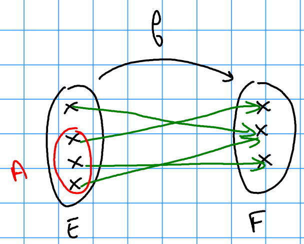
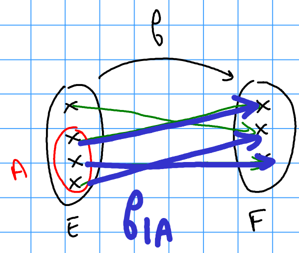
ALG 1 — **Chap 2 : Les applications** — Cours 4 (1)

2.1. Applications

2.1.2. Restrictions et prolongements

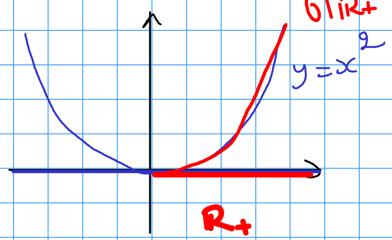


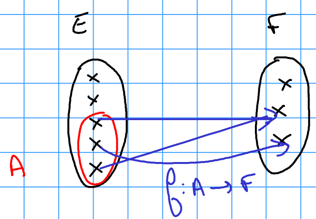


$f_{/A}$: f restreint à A.

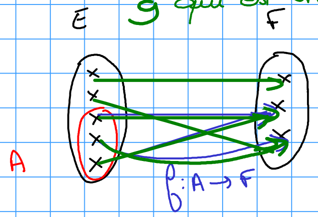
↑ se dit "restreint à"

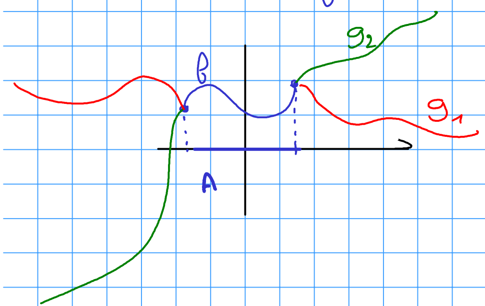
$f_{|\mathbb{R}_+}$ où $f : x \longmapsto x^2$





g qui est un prolongement de f



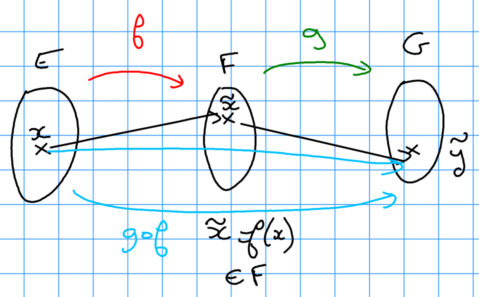


Ex 14 $\tilde{f}$ est un prolongement de f :

pour tout $x \in \underbrace{\mathbb{R}^*}_{\text{ens. de déf. de } f}$ on a $\tilde{f}(x) = f(x)$.

$$\frac{\sin(x)}{x} \xrightarrow[x \to 0]{} 1 \; : \; \frac{\sin(x)}{x} = \frac{\sin(x) - \sin(0)}{x - 0} \xrightarrow[x \to 0]{} \sin'(0) = \cos(0) = 1 .$$

$g \circ f$: se dit "g rond f"



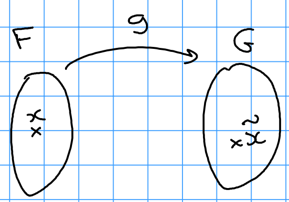
$g \circ f(x) = \tilde{y}$

$\tilde{y} = g(\tilde{x})$

et $\tilde{x} = f(x)$,

donc $\tilde{y} = g(f(x))$

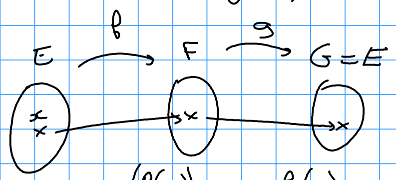
$g \circ f(x) = g(f(x))$.



$f \circ g$?

$g : F \to G$

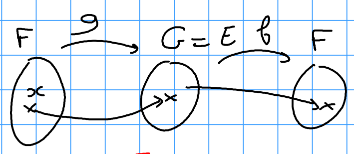
$f(\tilde{x})$? $\quad f : E \to F$ et $\tilde{x} \in G$



$g(\underbrace{f(x)}_{\in E}) = g \circ f(x)$

est bien défini

$\in F$



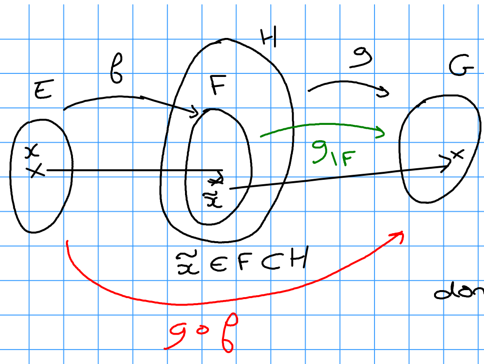
$f \circ g(\overset{\in F}{x}) = f(\underbrace{g(x)}_{\in E \text{ car } G = E})$ est bien défini.

<u>Remarque</u> $\times$ dans $\mathbb{R}$ est commutative : $2 \times 3 = 3 \times 2$

$\circ$ n'est pas commutative : $f \circ g \neq g \circ f$.

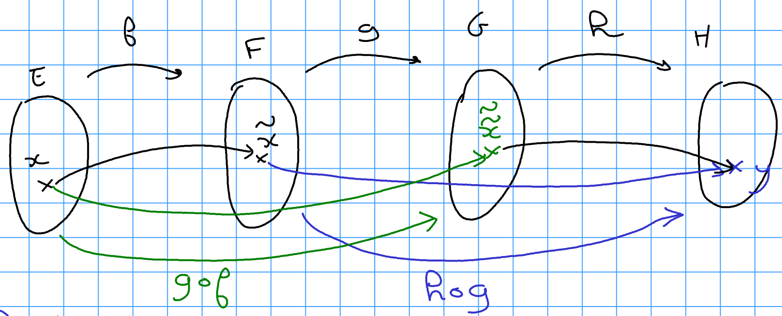
$g(\tilde{x}) = \tilde{x} + 1 = f(x) + 1$

$g \circ f(1) = 2$ et $f \circ g(1) = 4$ donc $g \circ f(1) \neq f \circ g(1)$. Donc $g \circ f \neq f \circ g$.



$\tilde{x} \in F \subset H$

donc $\tilde{x} \in H$



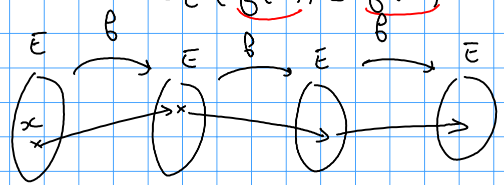
$(h \circ g) \circ f$

$h \circ (g \circ f)$

$(h \circ g) \circ f = h \circ (g \circ f)$.

$\forall x \in E, \quad id_E(x) = x$

$id_E(f(x)) = f(x)$



$f \circ f \circ \dots \circ f = f^n$ (f itéré n fois)

Dans $\mathbb{R}$, $(ab)^n = a^n b^n$

Ici, ce n'est plus le cas.

On suppose que $f \circ g = g \circ f$.

$$
\begin{aligned}
(f \circ g)^n &= (f \circ g) \circ (f \circ g) \circ (f \circ g) \circ \dots \circ (f \circ g) \\
&= f \circ g \circ f \circ g \circ f \circ g \circ \dots \circ f \circ g \\
&= f \circ f \circ \dots \circ f \circ g \circ g \circ \dots \circ g \\
&= f^n \circ g^n.
\end{aligned}
$$

car $g \circ f = f \circ g$

Ex 20 $\quad f(x) = x+1$

$$f^2(x) = (f\circ f)(x) = f(\underbrace{f(x)}_{\tilde{x}}) = \underbrace{f(x)}_{\tilde{x}}+1 = x+1+1 = x+2.$$

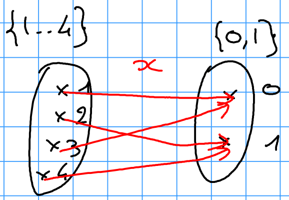
$$f^3(x) = (f\circ \underbrace{f\circ f}_{f^2})(x) = (f\circ f^2)(x) = f(\underbrace{f^2(x)}_{\tilde{x}}) = \underbrace{f^2(x)}_{\tilde{x}}+1 = x+2+1 = x+3.$$

$f^n(x) = x+n$. On peut le démontrer par récurrence.

$$(g\circ f)^2(x) = ((g\circ f)\circ(g\circ f))(x)$$
$$= (g\circ f)((g\circ f)(x))$$
$$= (g\circ f)(\underbrace{x^2+1}_{\tilde{x}})$$
$$= (x^2+1)^2+1$$

$(x_1, \dots x_n)$ correspond à l'application

$$x : \{1..n\} \longrightarrow E$$
$$i \longmapsto x_i$$



$$x : \{1..4\} \to \{0,1\} \text{ telle que } \dots$$
$$i \longmapsto x_i$$

$$(0, 1, 0, 1) = (x_i)_{i\in\{1..4\}}$$

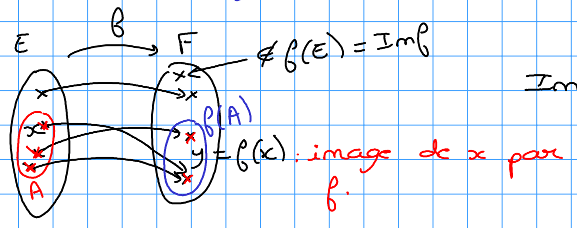
$(u_n)_{n\in\mathbb{N}} \longrightarrow$ $\quad u : \mathbb{N} \longrightarrow \mathbb{R}$, $\quad n \longmapsto u_n$.

La suite $(u_n)_{n\in\mathbb{N}}$ telle que $u_n = n^2$.

$$u : \mathbb{N} \longrightarrow \mathbb{R}$$
$$n \longmapsto n^2$$

2.2. Image directe, image réciproque

2.2.1. Image directe

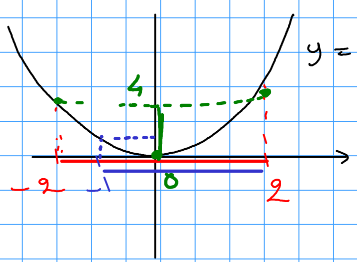


$\mathrm{Im} f = f(E) = \{ y \in F, \exists x \in E \quad y = f(x) \}$
$\subset F$

Ex 26 • $\mathrm{Im}\, id_E = E$. $\quad id_E : E \to E$, $x \mapsto x$.

$\mathrm{Im}\, id_E = id_E(E) = \{ id_E(x), x \in E \} = \{ x, x \in E \} = E$.

• $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$



$f([-2,2]) = \{ f(x), x \in [-2,2] \}$
$= \{ x^2, x \in [-2,2] \}$
$= [0,4]$.

x	$-\infty$	-2 -1	0	2	$+\infty$
f	$+\infty$ ↘	4 1	0	↗ 4	$+\infty$

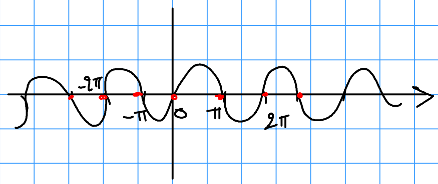
$f([-1,2]) = \{ x^2, x \in [-1,2] \} = [0,4]$

$\mathrm{Im} f = \{ x^2, x \in \mathbb{R} \} = \mathbb{R}_+$.

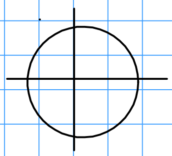
- $\mathrm{Im} f \subset \mathbb{R}_+$
- Soit $y \in \mathbb{R}_+$. Posons $x = \sqrt{y} \in \mathbb{R}$ alors $x^2 = y$
  donc $y \in \mathrm{Im} f$. Donc $\mathbb{R}_+ \subset \mathrm{Im} f$.

$\mathrm{Im} f = \{ \mathrm{Im}(z)^2, z \in \mathbb{C} \} \subset \mathbb{R}_+$

On a pris $x \in \mathbb{R}_+$. On a montré qu'il existe $z \in \mathbb{C}$ tel que
$x = f(z) \in \mathrm{Im} f = \{ f(z), z \in \mathbb{C} \}$.
Donc $\mathbb{R}_+ \subset \mathrm{Im} f$.



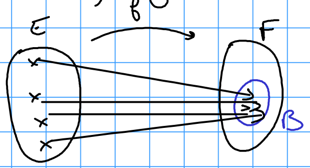
$$\sin(\pi\mathbb{Z}) = \{\sin(x),\ x \in \pi\mathbb{Z}\}$$
$$= \{\sin(\pi k),\ k \in \mathbb{Z}\}$$
$$= \{0\}.$$



$$\mathrm{Im}([0,2\pi]) = \{\sin(x),\ x \in [0,2\pi]\}$$
$$= [-1,1].$$

$\mathrm{Im}\, f = \{f(x),\ x \in E\} \subset B$

$\forall x \in E,\ f(x) \in B.$



x	$-\infty$		-1		0		3		$+\infty$
f	$+\infty$	$\searrow$	1	$\searrow$	0	$\nearrow$	9	$\nearrow$	$+\infty$

$f(\complement A) = [1,+\infty[\ \cup\ [9,+\infty[\ = [1,+\infty[$

$f(A) = [0,9]$

$f([-1,1]) = [0,1] \subset [-1,1].$