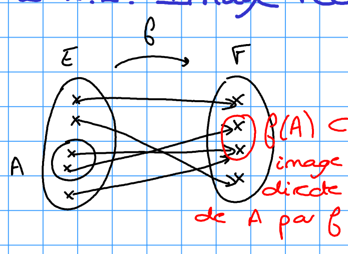
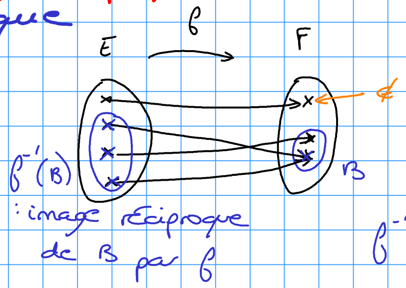
ALG 1 — **Chap 2 : Les applications** — Cours 5 (1)

2.2. Image directe, image réciproque

2.2.2. Image réciproque





$f^{-1}(B) \subset E$

On a $E = \mathbb{R}$

<u>Rem 36</u> • $f^{-1}(\{y\}) = \{x \in \cancel{E}^{\mathbb{R}},\ f(x) \in \{y\}\}$

$= \{x \in \cancel{E}^{\mathbb{R}},\ f(x) = y\}$

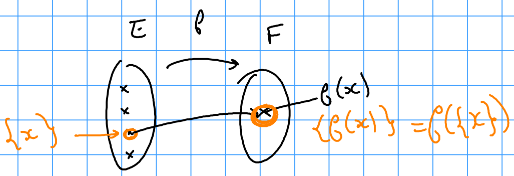
$f^{-1}(B) = \{x \in E,\ f(x) \in B\}$

$f(x) \in \{y\}$

$\Leftrightarrow\ f(x) = y$

• $f^{-1}([a,b]) = \{x \in \cancel{E}^{\mathbb{R}},\ f(x) \in [a,b]\}$

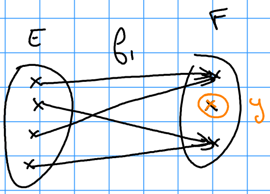
$= \{x \in \cancel{E}^{\mathbb{R}},\ a \leq f(x) \leq b\}$



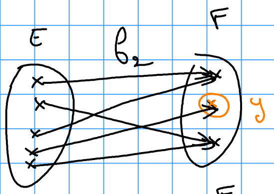
$f(\{x\}) = \{f(\tilde{x}),\ \overbrace{\tilde{x} \in \{x\}}^{\tilde{x} = x}\}$

$= \{f(x)\}$.

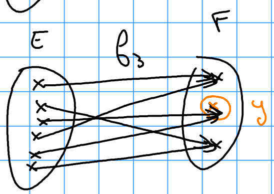
$f(A) = \{f(\tilde{x}),\ \tilde{x} \in A\}$



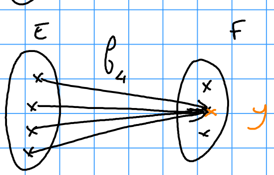
$f_1^{-1}(\{y\}) = \emptyset$



$f^{-1}(\{y\})$ a un élément.

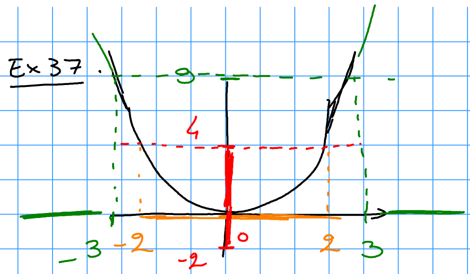


$f_3^{-1}(\{y\})$ a 2 éléments.



$f^{-1}(\{y\}) = E$

Ex 37.



$$f^{-1}([0,4]) = \{x \in \mathbb{R},\ f(x) \in [0,4]\}$$
$$= \{x \in \mathbb{R},\ 0 \leq x^2 \leq 4\}$$
$$= [-2,2]$$

x	$-\infty$	-3	-2		0		2	3	$+\infty$
$f(x)$	$+\infty$	9	4	$\searrow$	0	$\nearrow$	4	9	$+\infty$

$$f^{-1}([-2,4]) = \{x \in \mathbb{R},\ -2 \leq x^2 \leq 4\}$$
$$= \{x \in \mathbb{R},\ 0 \leq x^2 \leq 4\} = [-2,2]$$

$$f^{-1}([9,+\infty[) =]-\infty,3] \cup [3,+\infty[.$$

$$f^{-1}([-2,-1]) = \{x \in \mathbb{R},\ -2 \leq x^2 \leq -1\} \qquad x^2 \geq 0$$
$$= \emptyset.$$

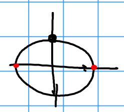
- $\exp^{-1}(\mathbb{R}_+) = \{x \in \mathbb{R},\ \exp(x) \in \mathbb{R}_+\} \qquad \forall x \in \mathbb{R},\ \exp(x) > 0$

  $= \mathbb{R}$

- $\sin^{-1}(\{1\}) = \{x \in \mathbb{R},\ \sin(x) = 1\}$

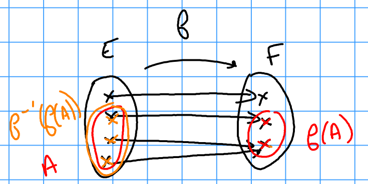
  $= \left\{\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}\right\}$

  $= \frac{\pi}{2} + 2\pi\mathbb{Z}.$

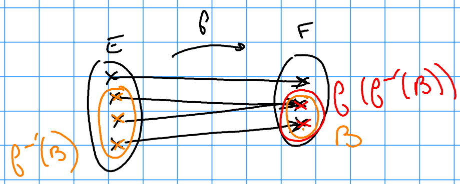


$$\sin^{-1}(\{2\}) = \{x \in \mathbb{R},\ \sin(x) = 2\} \qquad \forall x \in \mathbb{R}\ \ -1 \leq \sin(x) \leq 1$$
$$= \emptyset$$

$$\sin^{-1}([0,1]) = \{x \in \mathbb{R},\ 0 \leq \sin(x) \leq 1\}$$
$$= \bigcup_{k \in \mathbb{Z}} [2k\pi,(2k+1)\pi] \qquad \left([0,\pi] \cup [2\pi,3\pi]\right).$$



$f^{-1}(f(A))$

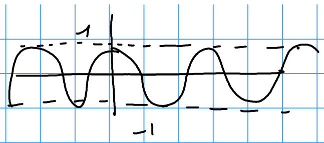


$f(f^{-1}(B))$

$\forall x \in \mathbb{R},\ \cos(x) \in [-1,1].$

$$f(x) = \cos(x), \qquad \cos^{-1}(\underbrace{\cos([0,2\pi])}_{=[-1,1]}) = \cos^{-1}([-1,1])$$
$$= \{x \in \mathbb{R},\ \cos(x) \in [-1,1]\}$$
$$= \mathbb{R}$$

$[0,2\pi] \subset \cos^{-1}(\cos([0,2\pi]))$
$\not\supset$



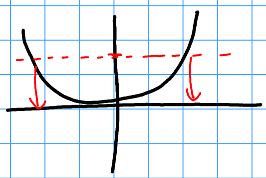
$\cos(\underbrace{\cos^{-1}(\mathbb{R})}) = \cos(\mathbb{R}) = [-1,1] \quad \subset \mathbb{R}$
$\not\supset$

$= \{x \in \mathbb{R},\ \cos(x) \in \mathbb{R}\}$
$= \mathbb{R}$

2.3. Injectivité, surjectivité, bijectivité

Cours 5 (2)

2.3.1. Injectivité



2 antécédents.

- Soit $z_1, z_2 \in \mathbb{C}\setminus\{1\}$. Supposons que $g(z_1) = g(z_2)$. Montrons que $z_1 = z_2$.

Alors $\frac{z_1+i}{z_1-1} = \frac{z_2+i}{z_2-1}$. Donc $(z_1+i)(z_2-1) = (z_2+i)(z_1-1)$

donc $\cancel{z_1z_2} - z_1 + iz_2 \cancel{-i} = \cancel{z_2z_1} - z_2 + iz_1 \cancel{-i}$

donc $z_2(i+1) = z_1(i+1)$

donc $z_2 = z_1$ (car $i+1 \neq 0$).

Donc g est injective.

- $id_E : E \to E$. Soit $(u,v) \in E^2$. Supposons que $id_E(u) = id_E(v)$.

Alors $u = v$. Donc id_E est injective.

2.3.2. Surjectivité

Cours 5 (3)

$f : E \to F$, $f(E) = \mathrm{Im} f \subset F$

$\tilde{f} : E \to \mathrm{Im} f \ (\subset F)$ $\quad \forall x \in E,\ \tilde{f}(x) = f(x) \in \mathrm{Im} f$
$\quad x \mapsto f(x)$

$\tilde{f}$ est surjective : Soit $y \in \mathrm{Im} f$. Il existe $x \in E$ tel que $y = f(x)$
$= \tilde{f}(x)$.

- $\mathrm{Im} f = \{x^2, x \in \mathbb{R}\} = \mathbb{R}_+ \neq \mathbb{R}$ donc f n'est pas surjective.
- $g : \mathbb{R} \to \mathbb{U}$ $\qquad (|e^{i\theta}| = 1$ donc $e^{i\theta} \in \mathbb{U})$.
  $\theta \mapsto e^{i\theta}$

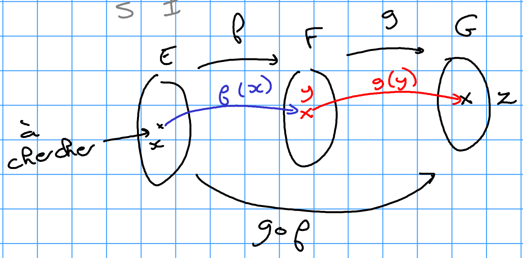
Soit $z \in \mathbb{U}$. Il existe $\theta \in \mathbb{R}$ tel que $z = e^{i\theta} = g(\theta)$.

Donc z admet au moins un antécédent donc g est surjective.

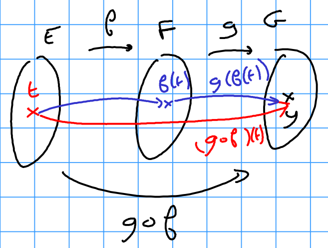
- $\mathrm{Im}\ id_E = id_E(E) = E$ $\quad (= \{id_E(x), x \in E\} = \{x, x \in E\} = E)$.

$g\circ f$

S I



$$(g\circ f)(x) = g(f(x)) \\ = g(y) \\ = z.$$



$(g\circ f)(x) = g(e^x - 1) = (e^x - 1)^2$. $\quad g\circ f : \mathbb{R} \to \underline{\mathbb{R}_+}$

Soit $y \in \mathbb{R}_+$. On cherche $x \in \mathbb{R}$ tel que $y = (e^x - 1)^2$.

Alors $e^x - 1 = \sqrt{y}$ donc $e^x = \sqrt{y} + 1 > 0$

donc $x = \ln(\sqrt{y} + 1) \in \mathbb{R}$.

Donc $y = (g\circ f)(x)$