Ex 1 $\quad GL_n(\mathbb{R})$ est dense dans $\mathcal{M}_n(\mathbb{R})$.

$P(x) = \det(A + x I_n) \qquad A \in GL_n(\mathbb{R})$.

$A \in \mathcal{M}_n(\mathbb{R})$.

$$\det B = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod b_{i,\sigma(i)}.$$

$$\det(A + x I_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} \left(a_{i,\sigma(i)} + x \delta_{i,\sigma(i)}\right)$$

$= P(x)$ polynôme de degré n.

P admet au plus n racines $\alpha_1 \dots \alpha_n$.

$\forall i \quad \alpha_i \neq 0 \quad$ car $\quad P(0) = \det(A) \neq 0$.

$0 < \varepsilon < \min_{i \in [\![1,n]\!]} (|\alpha_i|)$, $\alpha_i \neq 0$. $\quad \forall x, \quad |x| < \varepsilon \Rightarrow P(\varepsilon) \neq 0$.

$x \neq 0$.

$\Rightarrow \quad A + \varepsilon I_n \in GL_n(\mathbb{R})$

$\forall A \in \mathcal{M}_n(\mathbb{R}) \quad , \quad \forall \varepsilon > 0$

$B(A, \varepsilon) \cap GL_n(\mathbb{R}) \neq \emptyset$

$A + t I_n \quad |t| < \varepsilon$

$A + \frac{1}{k} I_n \longrightarrow A \qquad k >> 0 \quad A + \frac{1}{k} I_n \in GL_n(\mathbb{R})$

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2) $AB = BA \Rightarrow \mathrm{com}A \times \mathrm{com}B = \mathrm{com}B \times \mathrm{com}A$?

$\forall A \in \mathcal{M}_n(\mathbb{R}) \qquad A\,{}^t\mathrm{com}(A) = {}^t\mathrm{com}(A)\,A = \det A \, I_n$

Si $A \in GL_n(\mathbb{R})$ $\frac{1}{\det A} {}^t\mathrm{com}(A) = A^{-1}$

On sait dans ce cas que

$AB = BA \Rightarrow A^{-1}B^{-1} = B^{-1}A^{-1}$

$\Rightarrow {}^t\mathrm{com}(A)\, {}^t\mathrm{com}(B) = {}^t\mathrm{com}(B)\, {}^t\mathrm{com}(A)$

$\Rightarrow \mathrm{com}\, A\, \mathrm{com}\, B = \mathrm{com}\, B\, \mathrm{com}\, A$

Si A qcq $\exists (A_k) \in (GL_n(\mathbb{R}))^{\mathbb{N}}$ cv vers A

B $\exists (B_k)$ cv vers B

$A_k = A + \frac{1}{k} I_n$ $B_k = B + \frac{1}{k} I_n$

Si $AB = BA$, $\forall k \geq 0$ $\left(A + \frac{1}{k} I_n\right)\left(B + \frac{1}{k} I_n\right)$
$$= \left(B + \frac{1}{k} I_n\right)\left(A + \frac{1}{k} I_n\right)$$

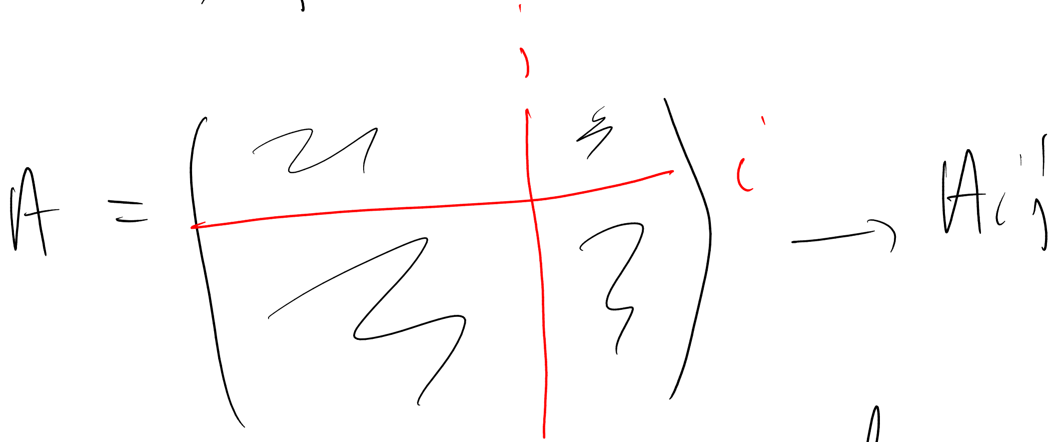
$$\operatorname{Com}\left(A + \frac{1}{k} I_n\right) \operatorname{Com}\left(B + \frac{1}{k} I_n\right) =$$
$$\operatorname{Com}\left(B + \frac{1}{k} I_n\right) \operatorname{Com}\left(A + \frac{1}{k} I_n\right)$$

"On passe à la limite"

$$\operatorname{Com} A \operatorname{Com} B = \operatorname{Com} B \operatorname{Com} A$$

L'application $\operatorname{Com} : \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathcal{M}_n(\mathbb{R})$
$$A \longmapsto \operatorname{Com} A$$

$$(\mathrm{Com}\, A)_{ij} = (-1)^{i+j} \det A_{ij}$$



Com est continue car les applications coordonnées sont continues :
$\det A_{ij}$ est un polynôme en les coefficients de la matrice A

On en déduit le résultat.

<u>Ex 2</u> : $A, B \in \mathcal{M}_n(\mathbb{Z})$: $\det A \wedge \det B = 1$

$\Rightarrow \exists u, v \in \mathbb{Z}$ tq

$$u \det A + v \det B = 1.$$

On pose $U = u \cdot {}^t\mathrm{com} A$ $\quad V = v \cdot {}^t\mathrm{com} B$

$$\begin{aligned} UA + VB &= u\, {}^t\mathrm{com} A\, A + v\, {}^t\mathrm{com} B\, B \\ &= u \det A\, I_n + v \det B\, I_n \\ &= I_n. \end{aligned}$$

Ex 3

$$f = \det \begin{pmatrix} a+\lambda_1 & & a \\ & & \\ a & & a+\lambda_n \end{pmatrix} = \det(aV+\lambda_1 E_1, \quad , aV+\lambda_n E_n)$$

$$V = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad E_n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$f \underset{\text{alternée}}{\overset{n \text{ linéaire}}{=}} \det(aV, \lambda_2 E_2 \quad , \lambda_n E_n) + \cdots$ $(*_i)$

$+ \det(\lambda_1 E_1, \quad \lambda_{n-1} E_{n-1}, aV)$

$+ \det(\lambda_1 E_1, \quad , \lambda_n E_n)$,

$= \lambda_1 \cdots \lambda_n$, $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$,

$$(\ast_i) = \det\left(\lambda_1 E_1, \quad \underset{i}{aV}, \quad \lambda_n E_n\right)$$

$$= \det \begin{array}{c} \\ \\ \\ \end{array}\begin{pmatrix} \lambda_1 & & & a & & & \\ & \ddots & & \vdots & & & \\ & & \lambda_{i-1} & \vdots & & & \\ & & & \vdots & \lambda_{i+1} & & \\ & & & \vdots & & \ddots & \\ & & & a & & & \lambda_n \end{pmatrix} \leftarrow i$$

$$= a \begin{vmatrix} \lambda_1 & & & & 0 \\ & \lambda_{i-1} & & & \\ & & \lambda_{i+1} & & \\ & & & & \\ 0 & & & & \lambda_n \end{vmatrix} = a \lambda_1 \hat{\lambda_i} \lambda_n$$

$$\det H = \prod_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} a \prod_{\substack{j=1 \\ j \neq i}}^{n} \lambda_j .$$

Ex4 $\quad n \geq 2 \quad A \in \mathcal{M}_n(\mathbb{K})$

$rg(A) = n \qquad rg\, Com(A) = n$

car $\quad A\ {}^t com(A) = \det A\, I_n \quad$ si $rg\, A = n$, $\det A \neq 0$

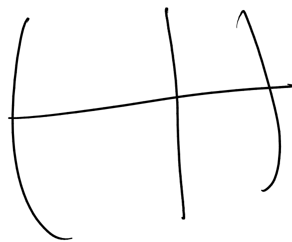
$\Rightarrow \dfrac{1}{\det A}\ {}^t Com(A) = A^{-1}$

$\Rightarrow {}^t Com\, A$ inversible $\Rightarrow Com\, A$ inversible

$\Rightarrow rg\, Com(A)$

$rg\, A \leq n-2 \qquad \underline{Com\, A = (0)}$

$(Com\, A)_{ij} = \det \underset{n-1}{A_{ij}}$



j

i

On en déduit que $\det A_{i_0 j} = 0$.

Si $\operatorname{rg} A = n-1$

$$A = \begin{array}{c} i_0 \\ \left(\quad \Big| \quad \right) \end{array} n$$

$$n$$

$(C_1 \; C_{j_0-1} \; C_{j_0+1} \; \cdots \; C_n)$ libre.

$$A' = \begin{pmatrix} C_1 & \widehat{C_{j_0}} & C_n \end{pmatrix} \quad n$$

$$n-1$$

$${}^tA' = \begin{pmatrix} {}^tC_1 \\ \widehat{{}^tC_{j_0}} \\ {}^tC_n \end{pmatrix} \quad n-1$$

$$n$$

$\mathrm{rg}\, {}^tA' = n-1$ $\qquad$ $${}^tA' = \begin{array}{c} \begin{array}{ccc} & i_0 & \end{array} \\ \begin{pmatrix} C'_1 & \cdots & C'_n \end{pmatrix} \end{array}$$

avec $\begin{pmatrix} C'_1 & \widehat{C'_{i_0}} & C'_n \end{pmatrix}$ est libre ;

$$A'' = \begin{array}{cc} & j_0 \\ \begin{pmatrix} A & \\ & \end{pmatrix} & i_0 \end{array}$$

$\mathrm{rg}\, A'' = n-1$;

$(\mathrm{Com}(A''))_{i_0,j_0} \neq 0$;

$\Rightarrow \mathrm{Com}(A) \neq 0$ $\Rightarrow \mathrm{rg}\, \mathrm{Com}(A) \geq 1$;

Mais $\det A = 0 \Rightarrow A\, {}^t\mathrm{com}(A) = 0$

$\Rightarrow \mathrm{Im}\, {}^t\mathrm{com}\, A \subset \underbrace{\mathrm{Ker}\, A}$

$\dim \mathrm{Ker}\, A = 1$;
$\mathrm{rg}\, A = n-1$

$\dim \mathrm{Im}\, {}^t\mathrm{com}\, A = 1$

$\Rightarrow \mathrm{rg}\, \mathrm{com}(A) = 1$;

Ex 5 : A diagonalisable :

$\exists\, D$ diagonale tq $P^{-1}AP = D$;

$\Rightarrow {}^tD = {}^tP\, {}^tA\, ({}^tP)^{-1} \Rightarrow {}^tA$ diagonalisable ;

Ex6 : $A \in GL_n(\mathbb{K})$ $B \in \mathcal{M}_n(\mathbb{K})$:

Si AB diagonalisable, alors BA diagon^{ble} :

$$P^{-1}ABP = D$$

$$\underbrace{P^{-1}A}_{Q}\,BA\,\underbrace{A^{-1}P}_{Q^{-1}} = D \Rightarrow BA \text{ diagonali}^{ble}$$

Ex7 : $A = \begin{pmatrix} 1 & 3 & 6 \\ 3 & -2 & -1 \\ 0 & 1 & 1 \end{pmatrix}$;

$$\chi_A(\lambda) = \det(\lambda I_n - A)$$

$$= (\lambda - 1)(\lambda - 3)(\lambda + 4),$$

$$\ker(A - \lambda I_n) \qquad \lambda = 1, 3, -4.$$

$$\ker A - I_n = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \qquad \ker A - 3I_n = \mathbb{R}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$\ker A + 4I_n = \mathbb{R}\begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 2 & -5 \\ 3 & -1 & 1 \end{pmatrix} \qquad P A P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \chi_B(\lambda) = (\lambda - 1)^3$$

Si B diagonalisable $B = I_n$

$$P^{-1} B P = I_n \implies B = I_n$$

$$C = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

$\mathrm{rg}\, C = 1$

$\mathrm{Ker}\, C$ de dim $n-1$

$$\mathrm{Ker}\, C = \mathrm{vect}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ -1 \end{pmatrix} \right)$$

$n-1$

$$C\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = n\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

$$P = \begin{pmatrix}1 & 1 & & 1\\ \vdots & -1 & & 0\\ \vdots & 0 & & \vdots\\ \vdots & \vdots & & \vdots\\ 1 & 0 & & -1\end{pmatrix}$$

$$PCP^{-1} = \begin{pmatrix} n & 0 & & 0\\ & \ddots & & \\ 0 & & & 0\end{pmatrix}$$