EQUA. DIFF

Chap 3 . Equations différentielles linéaires vectorielles du premier ordre

Cours 11 (1)

2. Méthode de variation des constantes

2.2 . Application aux équations différentielles linéaires scalaires d'ordre 2

$Y' = A(t)Y + b(t)$, $A(t) \in \mathcal{M}_n(\mathbb{K})$, $b \in \mathcal{M}_{n,1}(\mathbb{K})$. $\varphi_i : I \to \mathbb{K}^n$

Si $(\varphi_1, \dots \varphi_n)$ un système fondamental de solutions de $Y' = A(t)Y$

alors $S_H = \{ I \to \mathbb{K} ;\ t \mapsto \lambda_1 \varphi_1(t) + \dots + \lambda_n \varphi_n(t),\ (\lambda_1, \dots \lambda_n) \in \mathbb{K}^n \}$.

On cherche φ_p sous la forme $\varphi_p(t) = \psi_1(t)\varphi_1(t) + \dots + \psi_n(t)\varphi_n(t)$

avec les $\psi_i : I \to \mathbb{K}$ dérivables.

Alors φ_p est solution ssi $\psi_1'(t)\varphi_1(t) + \dots + \psi_n'(t)\varphi_n(t) = b(t)$

$S = S_H + \varphi_p$.

$y'' + a(t)y' + b(t)y = c(t)$. $(*)$

Soit $\varphi : I \to \mathbb{K}$. Supposons φ solution

$$\forall t \in I,\quad \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}'(t) = \begin{pmatrix} \varphi'(t) \\ \varphi''(t) \end{pmatrix} = \begin{pmatrix} \varphi'(t) \\ -a(t)\varphi'(t) - b(t)\varphi(t) + c(t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -b(t) & -a(t) \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}(t) + \begin{pmatrix} 0 \\ c(t) \end{pmatrix}$$

Donc $\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}$ est solution de $Y' = \begin{pmatrix} 0 & 1 \\ -b(t) & -a(t) \end{pmatrix} Y + \begin{pmatrix} 0 \\ c(t) \end{pmatrix}$ $(**)$

Si $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ est une solution de $(**)$ alors φ_1 est solution $(*)$

<u>Rem 16.</u> $w_{\mathcal{B}c}(t) = \det_{\mathcal{B}c}\left(\begin{pmatrix} \varphi_1 \\ \varphi_1' \end{pmatrix}, \begin{pmatrix} \varphi_2 \\ \varphi_2' \end{pmatrix} \right) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = w(t)$

(φ_1, φ_2) système fondamental de $y'' + a(t)y' + b(t)y = 0$.

Soit $\varphi : I \to E$.

$$\begin{cases} \varphi = \psi_1 \varphi_1 + \psi_2 \varphi_2 \\ \varphi' = \psi_1 \varphi_1' + \psi_2 \varphi_2' \end{cases}$$

φ est solution de $(*) = c(t)$

ssi $\begin{cases} \psi_1' \varphi_1 + \psi_2' \varphi_2 = 0 \\ \psi_1' \varphi_1' + \psi_2' \varphi_2' = c \end{cases}$, $\psi_1' = \dfrac{\begin{vmatrix} 0 & \varphi_2 \\ c & \varphi_2' \end{vmatrix}}{\begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}} = \dfrac{\begin{vmatrix} 0 & \varphi_2 \\ c & \varphi_2' \end{vmatrix}}{\omega} = \dfrac{-\varphi_2 \times c}{\omega}$

$$\psi_2' = \frac{\begin{vmatrix} \varphi_1 & 0 \\ \varphi_1' & c \end{vmatrix}}{\omega} = \frac{\varphi_1 \times c}{\omega}$$

$$\varphi = \psi_1 \varphi_1 + \psi_2 \varphi_2 = (\rho_1 + \lambda_1)\varphi_1 + (\rho_2 + \lambda_2)\varphi_2$$
$$= \underbrace{\lambda_1 \varphi_1 + \lambda_2 \varphi_2}_{S_R} + \underbrace{\rho_1 \varphi_1 + \rho_2 \varphi_2}_{\varphi_p}$$

$\begin{cases} \varphi = \psi_1 \varphi_1 + \psi_2 \varphi_2 & (1) \\ \varphi' = \psi_1 \varphi_1' + \psi_2 \varphi_2' & (2) \end{cases}$ En dérivant (1), $\varphi' = \psi_1' \varphi_1 + \psi_1 \varphi_1' + \psi_2' \varphi_2 + \psi_2 \varphi_2' \quad (1')$

$(1') - (2)$: $0 = \psi_1' \varphi_1 + \psi_2' \varphi_2$

En dérivant (2), $\varphi'' = \psi_1' \varphi_1' + \psi_1 \varphi_1'' + \psi_2' \varphi_2' + \psi_2 \varphi_2''$.

Donc φ est solution de $y'' + a(t) y' + b(t) y = c(t)$

ssi $\psi_1' \varphi_1' + \psi_1 \varphi_1'' + \psi_2' \varphi_2' + \psi_2 \varphi_2'' + a(t) \psi_1 \varphi_1' + a(t) \psi_2 \varphi_2'$
$+ b(t) \psi_1 \varphi_1 + b(t) \psi_2 \varphi_2 = c(t)$.

ssi $\psi_1 (\underbrace{\varphi_1'' + a(t) \varphi_1' + b(t) \varphi_1}_{=0}) + \psi_2 (\underbrace{\varphi_2'' + a(t) \varphi_2' + b(t) \varphi_2}_{=0})$
$+ \psi_1' \varphi_1' + \psi_2' \varphi_2' = c(t)$

<u>Ex 19</u> . $y'' + y = 0$. $\quad x^2 + 1 \quad \Delta = -4 < 0$

$r_1 = i$ et $r_2 = -i$, $S_R = \{ t \mapsto \lambda_1 \underbrace{\cos(t)}_{\varphi_1(t)} + \lambda_2 \underbrace{\sin t}_{\varphi_2(t)}, (\lambda_1, \lambda_2) \in \mathbb{R}^2 \}$

$\alpha = 0$, $\beta = 1$.

(φ_1, φ_2) un système fondamental de solutions de $y'' + y = 0$.

On cherche φ_p sous la forme $\varphi_p(t) = \psi_1(t) \varphi_1(t) + \psi_2(t) \varphi_2(t)$

avec $\psi_1, \psi_2 : I \to \mathbb{R}$, dérivables <u>ET</u> $\varphi_p'(t) = \psi_1(t) \varphi_1'(t) + \psi_2(t) \varphi_2'(t)$.

Donc φ_p est solution de l'équation de $y'' + y = \frac{1}{\cos t}$ (E),

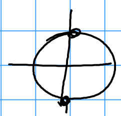
ssi $\begin{cases} \psi_1' \varphi_1 + \psi_2' \varphi_2 = 0 \\ \psi_1' \varphi_1' + \psi_2' \varphi_2' = \frac{1}{\cos} \end{cases}$

On a $\begin{cases} \varphi_1'(t)\cos(t) + \varphi_2'(t)\sin(t) = 0 \\ -\varphi_1'(t)\sin(t) + \varphi_2'(t)\cos(t) = \dfrac{1}{\cos(t)} \end{cases}$

$$\varphi_1'(t) = \frac{\begin{vmatrix} 0 & \sin t \\ \frac{1}{\cos t} & \cos t \end{vmatrix}}{\omega(t)} = \frac{-\sin(t)}{\cos(t) \times \omega(t)} = -\frac{\sin t}{\cos t}$$

$$\varphi_2'(t) = \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & \frac{1}{\cos t} \end{vmatrix}}{\omega(t)} = \frac{1}{\omega(t)} = 1 .$$

Or $\omega(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$



$$\int -\frac{\sin t}{\cos t}\,dt = \int \frac{du}{u} = \ln(|u|) = \ln(|\cos t|)$$
$u = \cos t$
$du = -\sin t$

$= \ln(\cos t)$ car $\cos(t) > 0$ sur I.

Prenons $\varphi_1(t) = \ln(\cos t)$, et $\varphi_2(t) = t$,

Alors $\varphi_p(t) = \ln(\cos t)\cos(t) + t\sin t$.

Donc $S = S_H + \varphi_p = \{ I \to \mathbb{R} ; t \mapsto \lambda_1 \cos t + \lambda_2 \sin t + \ln(\cos(t))\cos(t) + t\sin(t), (\lambda_1, \lambda_2) \in \mathbb{R}^2 \}$

Annexe : Utilisation des séries entières

Cours 11 (2)

$$y'' + a(t)y' + b(t)y = 0$$

$$y'' + 2ty' + 2y = 0 .$$

$f :]-R, R[\to \mathbb{R}$, $t \mapsto \sum_{n=0}^{+\infty} a_n t^n$.

Supposons $R > 0$. f est C^∞ sur $]-R, R[$.

Pour tout $t \in]-R, R[$, $f'(t) = \sum_{n=1}^{+\infty} a_n n t^{n-1}$, $f''(t) = \sum_{n=2}^{+\infty} a_n n(n-1) t^{n-2}$

Pour tout $t \in]-R, R[$,

$$f''(t) + 2t f'(t) + 2f(t) = \sum_{n=2}^{+\infty} a_n n(n-1)t^{n-2} + 2\sum_{n=1}^{+\infty} a_n n t^n + 2\sum_{n=0}^{+\infty} a_n t^n$$

$m = n-2$, $n = m+2$

$$= \sum_{m=0}^{+\infty} a_{m+2}(m+2)(m+1)t^m + 2\sum_{n=\cancel{1}\,0}^{+\infty} a_n n t^n + 2\sum_{n=0}^{+\infty} a_n t^n$$

$$= \sum_{n=0}^{+\infty} a_{n+2}(n+2)(n+1)t^n + 2a_n n t^n + 2a_n t^n$$

$$= \sum_{n=0}^{+\infty} \left(a_{n+2}(n+2)(n+1) + 2a_n(n+1)\right) t^n$$

f est solution de $y'' + 2ty' + 2y = 0$

ssi $\displaystyle\sum_{n=0}^{+\infty} \big(a_{n+2}(n+2)(n+1) + 2a_n(n+1)\big)t^n = 0 = \sum_{n=0}^{+\infty} 0\, t^n$

ssi $\forall n \in \mathbb{N}$, $a_{n+2}(n+2)(n+1) + 2a_n(n+1) = 0$

ssi $\forall n \in \mathbb{N}$, $a_{n+2} = \dfrac{-2a_n}{n+2}$.

ssi $\forall p \in \mathbb{N}$, $a_{2p} = \dfrac{-2}{2p} a_{2p-2} = \dfrac{-1}{p} a_{2p-2} = \dfrac{-1}{p} a_{2(p-1)}$

$$= \frac{-1}{p} \times \frac{-1}{p-1} a_{2(p-2)}$$

$$= \frac{-1}{p} \times \frac{-1}{p-1} \times \cdots \frac{-1}{1} a_0$$

$$= \frac{(-1)^p}{p!} a_0 .$$

$$a_{2p+1} = \frac{-2}{2p+1} a_{2p-1} = \frac{-2}{2p+1} \times \frac{-2}{2p-1} \times a_{2p-3}$$

$\times\, 2p \times (2p-2) \times \cdots \times 4 \times 2$

$$= \frac{-2}{2p+1} \times \frac{-2}{2p-1} \times \cdots \times \frac{-2}{3} \times a_1$$

$\times 2p \times \qquad \times\, 2p-2 \times \qquad \times 4 \times \qquad \times 2$

$$= \frac{(-2)^p \; 2\times p \times 2 \times (p-1) \times 2 \times (p-2) \times \cdots 2 \times 2 \times 2 \times 1}{(2p+1)!} a_1$$

$$= \frac{(-2)^p \times 2^p \; p!}{(2p+1)!} a_1 = \frac{(-1)^p 4^p p!}{(2p+1)!} a_1$$

• $u_p = \left| \dfrac{(-1)^p}{p!} t^{2p} \right|$ avec $t \neq 0$, $\dfrac{u_{p+1}}{u_p} = \dfrac{1}{p+1} |t|^2 \xrightarrow[p \to +\infty]{} 0$.

$$\sum_{p=0}^{+\infty} \frac{(-1)^p}{p!} t^{2p} = \sum_{p=0}^{+\infty} \frac{(-t^2)^p}{p!} = \exp(-t^2)$$

$\varphi_1(0) = 1$, $\varphi_2(0) = 0$

$\varphi_1'(0) = 0$ $\quad \varphi_2'(0) = 1$

$$w(0) = \begin{vmatrix} \varphi_1(0) & \varphi_2(0) \\ \varphi_1'(0) & \varphi_2'(0) \end{vmatrix} = 1 \neq 0$$

$f(t) = \sum a_n t^n$

↳ par réc, on a l'expression des a_n ; puis étude réciproque pour vérifier que la série obtenue a un rayon $R > 0$.