Ex1 : $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ $\quad y' = AX \quad \begin{pmatrix} 0 & & \\ & & * \end{pmatrix}$

$A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}^t \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ $\quad \ker A = \mathrm{vect}\left(\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$

$\mathrm{rg}\, A = \underline{1}$.

$E_0(A) = \mathrm{vect}\left(\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$.

$A\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}^t \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$E_6(A) = \mathrm{vect}\left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right)$.

$$y = \left\{ \lambda e^{0t}\begin{pmatrix}2\\-1\\0\end{pmatrix} + \mu e^{0t}\begin{pmatrix}1\\0\\1\end{pmatrix} + \gamma e^{6t}\begin{pmatrix}1\\2\\-1\end{pmatrix},\ \lambda,\mu,\gamma \in \mathbb{R} \right\}$$

<u>Ex2.</u>

$$y' = \begin{pmatrix}1 & 8\\ 2 & 1\end{pmatrix} y + \underbrace{\begin{pmatrix}e^{t}\\ e^{-3t}\end{pmatrix}}$$

$$A = \begin{pmatrix}1 & 8\\ 2 & 1\end{pmatrix} \qquad \chi_A = X^2 - 2X - 15. \qquad \delta = 1 + 6 = 4$$

$$= (X+3)(X-5)$$

A est diagonalisable.

$E_5 = \mathrm{vect}\left(\begin{pmatrix}2\\1\end{pmatrix}\right) \qquad E_{-3} = \mathrm{vect}\begin{pmatrix}-2\\1\end{pmatrix}.$

$$\boxed{\mathcal{Y}_0 = \left\{ \lambda e^{5t}\begin{pmatrix}2\\1\end{pmatrix} + \mu e^{-3t}\begin{pmatrix}-2\\1\end{pmatrix},\ \lambda,\mu\in\mathbb{R}\right\}}$$

$$P = \begin{pmatrix}2 & -2\\1 & 1\end{pmatrix} \qquad D = \begin{pmatrix}5 & 0\\0 & -3\end{pmatrix} = P^{-1}AP$$

$$Y = PZ \qquad B = \begin{pmatrix}e^{t}\\e^{-3t}\end{pmatrix} \qquad Y' = PZ'$$

$$Y' = AY + B \iff PZ' = APZ + B$$

$$\iff Z' = \underbrace{P^{-1}AP}_{=D}Z + P^{-1}B$$

$$\iff \begin{pmatrix}z_1'\\z_2'\end{pmatrix} = \begin{pmatrix}5 & 0\\0 & -3\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix} +$$

$$P = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$$

$$P^{-1} B = \frac{1}{4} \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} e^{t} \\ e^{-3t} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} e^{t} + e^{-3t} \\ -e^{-t} + 2e^{-3t} \end{pmatrix}$$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 5z_1 \\ -3z_2 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} e^{t} + e^{-3t} \\ e^{-t} + 2e^{-3t} \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \underbrace{\lambda_1 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mu e^{-3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{q_0} + \begin{pmatrix} -\frac{1}{8} e^{-3t} - t e^{-3t} \\ -\frac{1}{8} e^{t} - \frac{1}{16} e^{-3t} + \frac{1}{2} t e^{-3t} \end{pmatrix}$$

Variation des constantes $\psi = u_1 \varphi_1 + u_2 \varphi_2$:

$$\psi = \begin{array}{cc} \begin{pmatrix} 2e^{5t} & -2e^{-3t} \\ e^{5t} & e^{-3t} \end{pmatrix} \\ \varphi_1 \quad \varphi_2 \end{array} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2e^{5t} & -2e^{-3t} \\ e^{5t} & e^{-3t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} e^{t} \\ e^{-3t} \end{pmatrix}$$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{e^{-2t}}{4} \begin{pmatrix} e^{-3t} & 2e^{-3t} \\ -e^{5t} & 2e^{5t} \end{pmatrix} \begin{pmatrix} e^{t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} \frac{e^{-4t}}{4} + 2\frac{e^{-2t}}{4} \\ -\frac{e^{-4t}}{4} \quad \frac{e^{6t}}{2} \end{pmatrix} \square$$

Ex3

$$Y' = AY \qquad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad 1 \in \operatorname{Spec} A \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in E_1(A)$$

$$\chi_A = (X-1)(X^2+1) = (X-1)(X+i)(X-i)$$

A est diagonalisable dans $\mathbb{C}$.

$$A - iI_3 = \begin{pmatrix} 1-i & 0 & 1 \\ 0 & -1-i & -1 \\ \cancel{0} & \cancel{2} & \cancel{1-i} \end{pmatrix} \begin{pmatrix} 1 \\ -i \\ i-1 \end{pmatrix} \qquad E_i(A) = \operatorname{vect}\begin{pmatrix} 1 \\ -i \\ i-1 \end{pmatrix}$$

$rg = 2$

Rappel : $AZ = \lambda Z$ $\quad \overline{AZ} = A\bar{Z} = \bar{\lambda}\bar{Z}$

A réel

$$E_{-i}(A) = \mathrm{vect}\left(\begin{pmatrix}1\\ i\\ -i-1\end{pmatrix}\right)$$

$$\mathcal{S}_{\mathbb{C}} = \left\{ \lambda e^{t}\begin{pmatrix}1\\0\\0\end{pmatrix} + \mu \underbrace{e^{it}\begin{pmatrix}1\\ -i\\ i-1\end{pmatrix}}_{\varphi} + \gamma e^{-it}\begin{pmatrix}1\\ i\\ -i-1\end{pmatrix},\ \lambda,\mu,\gamma \in \mathbb{C} \right\}$$

$$\mathrm{Re}\,\varphi = \begin{pmatrix}\cos t\\ \sin t\\ -\cos t - \sin t\end{pmatrix}$$

$$\mathrm{Im}\,\varphi = \begin{pmatrix}\sin t\\ -\cos t\\ \cos t - \sin t\end{pmatrix}$$

$$\mathcal{S}_{\mathbb{R}} = \left\{ \lambda e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} \cos t \\ \sin t \\ -\cos t - \sin t \end{pmatrix} + \gamma \begin{pmatrix} \sin t \\ -\cos t \\ \cos t - \sin t \end{pmatrix} \;\middle|\; \lambda, \mu, \gamma \in \mathbb{R} \right\}$$

Ex4 $\begin{cases} x_1' = (2-t)x_1 + (t-1)x_2 \\ x_2' = 2(1-t)x_1 + (2t-1)x_2 \end{cases}$ $\quad X' = A(t)X$

$$A = \begin{pmatrix} 2-t & t-1 \\ 2(1-t) & 2t-1 \end{pmatrix} \qquad \chi_A = X^2 - (t+1)X + t = (X-1)(X-t)$$

$$E_1(A(t)) = \mathrm{vect}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \qquad E_t(A(t)) = \mathrm{vect}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$

$$P = \begin{array}{cc} v_1 & v_2 \\ \end{array}$$
$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad Y = PZ \qquad Y' = PZ'$$

P a coefficient constant.

$$Y' = A(t)\,Y \qquad PZ' = A(t)\,P\,Z$$

$$Z' = P^{-1} A(t)\, P\, Z$$

$$Z' = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} Z$$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} z_1 \\ t z_2 \end{pmatrix}$$

$$z_1 = \lambda e^{t}$$
$$z_2 = \mu\, e^{\frac{t^2}{2}}$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda e^{t} \\ \mu e^{\frac{t^2}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda e^{t} + \mu\, e^{\frac{t^2}{2}} \\ \lambda e^{t} + 2\mu\, e^{\frac{t^2}{2}} \end{pmatrix}$$

$\lambda, \mu \in \mathbb{R}$.

Remarque : les solutions ne sont pas

$$\lambda e^{t}\begin{pmatrix}1\\1\end{pmatrix} + \mu e^{t^2}\begin{pmatrix}1\\2\end{pmatrix}. \quad ⚠$$

Ex 5 : $A = \begin{pmatrix}-1 & -4\\ -1 & 3\end{pmatrix}$. $\qquad Y' = AY$.

$\chi_A = X^2 - 2X + 1 = (X-1)^2$. $\qquad \mathrm{Spec}\, A = \{1\}$.

A n'est pas diagonalisable car sinon semblable à I_2 (c'est faux $A \neq I_2$).

$$A - I_2 = \begin{pmatrix}\cancel{-2} & \cancel{-4}\\ -1 & 2\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}$$

$$E_1(A) = \mathrm{vect}\Big(\underbrace{\begin{pmatrix}2\\1\end{pmatrix}}_{e_1}\Big)$$

$$\begin{array}{cc|c} Ae_1 & Ae_2 & \\ \end{array}$$

$$\begin{array}{c} \\ \end{array}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{array}{l} e_1 \\ e_2 \end{array} \qquad e_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Ae_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad P = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

On revient $Y' = AY$ $\quad Y = PZ$

P à coef. cst $\quad Y' = PZ'$

$$Y' = AY \iff PZ' = APZ$$

$$\iff Z' = P^{-1}APZ$$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\begin{cases} z_1' = z_1 - z_2 \\ z_2' = z_2 \end{cases} \Longrightarrow \begin{cases} z_1' = z_1 - \lambda e^t \\ z_2 = \lambda e^t \end{cases}$$

$$\Longrightarrow \begin{cases} z_1 = \mu e^t - 2\lambda t e^t \\ z_2 = \lambda e^t \end{cases}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \ldots$$