TD 3 - Algèbre 2

Exercice 1

$\deg(0) = -\infty$, Si P est constant, $\deg P' = -\infty$

1. Soit P de degré ≥ 1.

$\deg(P')^2 = 2\deg P' = 2\times(\deg P - 1)$

$\deg 4P = \deg P$.

Si P vérifie $P'^2 = 4P$ alors $2(\deg P - 1) = \deg P$,

donc $\deg P = 2$.

P s'écrit $P = aX^2 + bX + c$ avec $a \neq 0$.

et $P' = 2aX + b$.

On a $P'^2 = 4P$ ssi $4a^2X^2 + 4abX + b^2 = 4aX^2 + 4bX + 4c$

ssi $\begin{cases} 4a^2 = 4a \\ 4ab = 4b \\ b^2 = 4c \end{cases}$ ssi $\begin{cases} a = 1 \\ c = \frac{1}{4}b^2 \end{cases}$

ssi $P = X^2 + bX + \frac{1}{4}b^2$ avec $b \in \mathbb{K}$.

L'ensemble des solutions est $S = \{X^2 + bX + \frac{1}{4}b^2, b \in \mathbb{K}\} \cup \{0\}$.

2. $\underbrace{(X^2+1)P''}_{d^\circ n} - \underbrace{6P}_{d^\circ n} = 0$

$P = a_0 + a_1X + a_2X^2 + \dots + a_nX^n$, avec $a_n \neq 0$

$P' = a_1 + 2a_2X + \dots + na_nX^{n-1}$

$P'' = \underline{2a_2 + 6a_3X + \dots + n(n-1)a_nX^{n-2}}$.

$(X^2+1)P'' = (X^2+1)\,\underline{\;''\;} = \underbrace{\dots}_{d^\circ < n} + n(n-1)a_nX^n$

$(X^2+1)P'' - 6P = \underbrace{\dots}_{d^\circ < n} + n(n-1)a_nX^n - \underbrace{\dots}_{d^\circ < n} - 6a_nX^n$

$$= \underbrace{\cdots}_{d^\circ < n} + (n(n-1)-6)a_n X^n .$$

Si P est solution alors $(n(n-1)-6)a_n = 0$

Donc $n^2 - n - 6 = 0$

soit $(n+2)(n-3) = 0$ et $n \geq 0$

donc $n = 3$.

P s'écrit $P = aX^3 + bX^2 + cX + d$ avec $a \neq 0$.

$P' = 3aX^2 + 2bX + c$

$P'' = 6aX + 2b$.

Donc P est solution ssi

$$\cancel{6aX^3} + 2bX^2 + 6aX + 2b - \cancel{6aX^3} - 6bX^2 - 6cX - 6d = 0$$

ssi $\begin{cases} 2b - 6b = 0 \\ 6a - 6c = 0 \\ 2b - 6d = 0 \end{cases}$ ssi $\begin{cases} b = 0 \\ a = c \\ d = 0 \end{cases}$

ssi $P = aX^3 + aX = a(X^3 + X)$.

On ajoute : le polynôme nul

Exercice 2

$P_n - P_n' = X^n$.

$P_n = a_0 + a_1 X + \dots + a_n X^n = \sum_{k=0}^{n} a_k X^k$

$P_n' = a_1 + 2a_2 X + \dots + n a_n X^{n-1} = \sum_{k=1}^{n} k a_k X^{k-1}$

$a_0 + a_1 X + \dots + a_{n-1}X^{n-1} + a_n X^n - a_1 - 2a_2 X \cdots - n a_n X^{n-1} = 1 X^n$

$i = k-1$ $= \sum_{i=0}^{n-1} (i+1) a_{i+1} X^i$

$= \sum_{k=0}^{n-1} (k+1) a_{k+1} X^k$

$$\sum_{k=0}^{n} a_k X^k - \sum_{k=0}^{n-1} (k+1) a_{k+1} X^k = X^n$$

ou $a_n X^n + \sum_{k=0}^{n-1} (a_k - (k+1)a_{k+1}) X^k = X^n$

$\begin{cases} a_0 = a_1 \\ a_1 = 2a_2 \\ \vdots \\ a_{n-2} = (n-1)a_{n-1} \end{cases}$

$a_n = 1$ et $\forall k \in \{0..n-1\}$, $a_k = (k+1)a_{k+1}$.

$$\begin{cases} a_{n-1} = n a_n \\ a_n = 1 \end{cases} \qquad \begin{cases} a_n = 1 \\ a_{n-1} = n \\ a_{n-2} = n(n-1) = \dfrac{n!}{(n-2)!} \\ \vdots \\ a_1 = n(n-1)\cdots \times 3 \times 2 \\ a_0 = n! \end{cases} \qquad a_{k+1} = \frac{1}{k+1} a_k$$

$$a_{k+1} = \frac{1}{k+1} a_k = \frac{1}{k+1} \times \frac{1}{k} \times a_{k-1} = \frac{1}{k+1} \times \frac{1}{k} \times \frac{1}{k-1} a_{k-2}$$
$$= \cdots$$
$$= \frac{1}{k+1} \times \frac{1}{k} \times \cdots \times \frac{1}{1} a_0$$
$$= \frac{1}{(k+1)!} a_0 .$$

On a $a_n = \dfrac{1}{n!} a_0$ et $a_n = 1$, donc $a_0 = n! \times 1 = n!$

Donc pour tout $k \in \{0..n\}$, $a_k = \dfrac{1}{k!} \times n! = \dfrac{n!}{k!}$.

Donc $P = X^n + \dfrac{n!}{(n-1)!} X^{n-1} + \dfrac{n!}{(n-2)!} X^{n-2} + \cdots + n!$.

Exercice 4 $\quad A = (a_{ij})_{\substack{i=1..n \\ j=1..n}}$, $B = (b_{ij})_{\substack{i=1..n \\ j=1..n}}$

$AB = (c_{ij})_{\substack{i=1..n \\ j=1..n}} \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad \sigma(A) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} \in \mathbb{K}.$$

On a $\sigma(A) \cdot \Pi = \sigma(A) \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \sigma(A) & \cdots & \sigma(A) \\ \vdots & & \vdots \\ \sigma(A) & \cdots & \sigma(A) \end{pmatrix}$

Posons $P = \underbrace{\Pi A \Pi}_{C} = (p_{ij})_{\substack{i=1..n \\ j=1..n}}$ Montrons que $p_{ij} = \sigma(A)$

On a $C = (c_{ij})_{\substack{i=1..n \\ j=1..n}}$.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \underbrace{m_{kj}}_{=1} = \sum_{k=1}^{n} a_{ik} = \sum_{l=1}^{n} a_{il}$$

On a donc $p_{ij} = \sum_{k=1}^{n} \underbrace{m_{ik}}_{=1} c_{kj} = \sum_{k=1}^{n} c_{kj} = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl}$

$$\sigma(A) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} \in \mathbb{K}.$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$$
$$= \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} = p_{ij}.$$

Donc $\Pi A \Pi = \sigma(A).\Pi$.

<u>Exercice 5</u>.

$$B = A - I_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B^3 = B^2 \times B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O_3.$$

Pour tout $n \geq 3$, $B^n = O_3$.

$A = I_3 + B$.

Donc $A^n = (I_3 + B)^n$ (Binôme de Newton autorisé car I_3 et B commutent :

$I_3 \times B = B$ et $B \times I_3 = B$) $I_3 \times B = B \times I_3$)

Donc $A^n = \sum_{k=0}^{n} \binom{n}{k} B^k \underbrace{I_3^{n-k}}_{= I_3} = \sum_{k=0}^{n} \binom{n}{k} B^k$

$= \binom{n}{0} B^0 + \binom{n}{1} B^1 + \binom{n}{2} B^2 + \underbrace{\cancel{\binom{n}{3} B^3 + \cdots}}_{= 0_3}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + n \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{n(n-1)}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & n & n + \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ $\frac{n(n+1)}{2}$

$x^0 = 1$

$B^0 = I_n$

$A^0 = I_n$.

Exercice 3

Notons $n = \deg P$. Montrons $P(X+1) = \sum_{k=0}^{n} \frac{1}{k!} P^{(k)}(X)$.

On a $P(X+1) = \sum_{k=0}^{n} \frac{P^{(k)}(1)}{k!} X^k$. $(*)$

Soit $\varphi \in K_n[X]$. $\varphi(X) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(0)}{k!} X^k$. (Formule de Taylor polynomiale)

Si $\varphi(X) = P(X+1)$, on obtient $(*)$ puisque $\varphi^{(k)}(X) = P^{(k)}(X+1)$

Si $\varphi(X) = P^{(k)}(X)$, on obtient $P^{(k)}(X) = \sum_{i=0}^{n} \frac{\varphi^{(i)}(0)}{i!} X^i$
$(\varphi^{(i)}(X) = P^{(k+i)}(X))$
$$= \sum_{i=0}^{n} \frac{P^{(k+i)}(0)}{i!} X^i.$$

Donc $P^{(k)}(1) = \sum_{i=0}^{n} \frac{P^{(k+i)}(0)}{i!}$

Donc
$$P(X+1) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{i=0}^{n} \frac{P^{(k+i)}(0)}{i!} X^k$$
$$= \sum_{k=0}^{n} \sum_{i=0}^{n} \frac{P^{(k+i)}(0)}{k!\, i!} X^k$$
$$= \sum_{i=0}^{n} \sum_{k=0}^{n} \frac{P^{(k+i)}(0)}{k!\, i!} X^k$$
$$= \sum_{i=0}^{n} \frac{1}{i!} \underbrace{\sum_{k=0}^{n} \frac{P^{(k+i)}(0)}{k!} X^k}_{P^{(i)}(X)}$$
(cf. Formule de Taylor polynomiale pour $\varphi(X) = P^{(i)}(X)$)
$$= \sum_{i=0}^{n} \frac{1}{i!} P^{(i)}(X)$$
$$= \sum_{k=0}^{n} \frac{1}{k!} P^{(k)}(X) = \sum_{k=0}^{+\infty} \frac{1}{k!} P^{(k)}(X).$$

Exercice 6

$0 \leq d \leq c \leq b \leq a$

et $b + c \leq a + d$.

$$M^{n+1} = M M^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} a a_n + b c_n & a b_n + b d_n \\ c a_n + d c_n & c b_n + d d_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix},$$

Pour tout $n \in \mathbb{N}^*$, $\begin{cases} a_{n+1} = a a_n + b c_n \\ b_{n+1} = a b_n + b d_n \\ c_{n+1} = c a_n + d c_n \\ d_{n+1} = c b_n + d d_n \end{cases}$

$$a_{n+1} + d_{n+1} - (b_{n+1} + c_{n+1}) = a(a_n - b_n) + b(c_n - d_n) + c(b_n - a_n) + d(d_n - c_n)$$

$$= \underbrace{(a-c)}_{\geq 0}(a_n - b_n) + \underbrace{(b-d)}_{\geq 0}(c_n - d_n)$$

On a $M^n = M^{n-1} M$

$$= \begin{pmatrix} a_{n-1} & b_{n-1} \\ c_{n-1} & d_{n-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a a_{n-1} + c b_{n-1} & b a_{n-1} + d b_{n-1} \\ a c_{n-1} + c d_{n-1} & b c_{n-1} + d d_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

$M^1 = M$

Donc $\begin{cases} a_n = a a_{n-1} + c b_{n-1} \\ b_n = b a_{n-1} + d b_{n-1} \\ c_n = a c_{n-1} + c d_{n-1} \\ d_n = b c_{n-1} + d d_{n-1} \end{cases}$

$a_1, b_1, c_1, d_1 \geq 0$

On a donc $a_n - b_n = \underbrace{a_{n-1}}_{\geq 0} \underbrace{(a - b)}_{\geq 0} + \underbrace{b_{n-1}}_{\geq 0} \underbrace{(c - d)}_{\geq 0} \geq 0$

$$c_n - d_n = c_{n-1}(a - b) + d_{n-1}(c - d) \geq 0$$

$$a_{n+1}+d_{n+1}-(b_{n+1}+c_{n+1}) = a(a_n-b_n)+b(c_n-d_n)+c(b_n-a_n)+d(d_n-c_n)$$
$$= \underbrace{\underbrace{(a-c)}_{\geqslant 0}\underbrace{(a_n-b_n)}_{\geqslant 0}+\underbrace{(b-d)}_{\geqslant 0}\underbrace{(c_n-d_n)}_{\geqslant 0}}_{\geqslant 0}$$

Donc pour tout $n \in \mathbb{N}^*$, $\quad b_{n+1}+c_{n+1} \leq a_{n+1}+d_{n+1}$.

Donc pour tout $n \geqslant 2$, $\quad b_n + c_n \leq a_n + d_n$.

$$\omega^{(k-1)(\ell-1)}$$

$$\omega^{\ell-1}$$

Exercice 7

$$A = \begin{array}{c} \\ k=1 \rightarrow \end{array}\begin{array}{c} \ell=1 \\ \downarrow \\ \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \cdots & \omega^{n-1} \\ \vdots & \omega^2 & \omega^4 & & \\ \vdots & \vdots & & \ddots & \\ 1 & \omega^{n-1} & \cdots & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix}\end{array} \qquad \omega = e^{\frac{2i\pi}{n}},$$

$$^t\overline{A} = \begin{pmatrix} \overline{1} & \overline{1} & \cdots & \cdots & \overline{1} \\ \overline{1} & \overline{\omega}^1 & \overline{\omega}^2 & \cdots & \overline{\omega}^{n-1} \\ \vdots & \overline{\omega}^2 & \overline{\omega}^4 & & \\ \vdots & \vdots & & \ddots & \\ \overline{1} & \overline{\omega}^{n-1} & \cdots & \cdots & \overline{\omega}^{(n-1)(n-1)} \end{pmatrix} \qquad \overline{\omega} = e^{-\frac{2i\pi}{n}}$$

$$\overline{1} = 1$$

$$A\overline{A} = (c_{ij})_{\substack{i=1..n \\ j=1..n}} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & c_{ij} & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$$

$A = (a_{ij})_{\substack{i=1..n \\ j=1..n}}$, $\quad \overline{A} = (a'_{ij})_{\substack{i=1..n \\ j=1..n}}$.

On a
$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, a'_{kj}$$
$$= \sum_{k=1}^{n} a_{ik}\, \overline{a_{jk}}$$
$$= \sum_{k=1}^{n} \omega^{(i-1)(k-1)}\, \overline{\omega^{(j-1)(k-1)}}$$
$$= \sum_{k=1}^{n} e^{\frac{2i\pi}{n}\times(i-1)(k-1)} \times e^{-\frac{2i\pi}{n}(j-1)(k-1)}$$
$$= \sum_{k=1}^{n} e^{\frac{2i\pi}{n}(k-1)((i-1)-(j-1))}$$

$$= \sum_{k=1}^{n} e^{\frac{2i\pi}{n}(k-1)(i-j)}$$

$$= \sum_{k=1}^{n} \left(e^{\frac{2i\pi}{n}(i-j)}\right)^{(k-1)}$$

$p = k-1$

$$= \sum_{p=0}^{n-1} \left(e^{\frac{2i\pi}{n}(i-j)}\right)^{p} \quad \text{(somme géométrique)}$$

$$= \begin{cases} \dfrac{1 - e^{\frac{2i\pi}{\cancel{n}}\overbrace{(i-j)}^{\in \mathbb{N}} \times \cancel{n}}}{1 - e^{\frac{2i\pi}{n}(i-j)}} & \text{si } i \neq j \\ n & \text{si } i = j \end{cases}$$

$$= \begin{cases} 0 & \text{si } i \neq j \\ n & \text{si } i = j \end{cases}$$

$$A\overline{A} = \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n \end{pmatrix} = n I_n$$