I. Structure d'espace vectoriel (ev)

3. Règles de calcul.

$\underset{\in \mathbb{R}}{0}\begin{pmatrix}1\\2\end{pmatrix} = \underbrace{\begin{pmatrix}0\\0\end{pmatrix}}_{\in \mathbb{R}^2}$, $\lambda\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, $\vec{x} = \begin{pmatrix}2\\3\end{pmatrix}$, $\overrightarrow{-x}$

Dans le groupe $(\mathbb{R}^2, +)$, $\begin{pmatrix}2\\3\end{pmatrix} + \overrightarrow{-x} = \begin{pmatrix}0\\0\end{pmatrix}$, $\overrightarrow{-x} = \begin{pmatrix}-2\\-3\end{pmatrix}$.

$\overrightarrow{-x} = -1\begin{pmatrix}2\\3\end{pmatrix} = -1.\vec{x}$.

Prop 13.

$\cancel{x} + y = \cancel{x} \Rightarrow y = e$

- $0.\vec{x} = (0+0).\vec{x} = 0.\vec{x} + 0.\vec{x}$, donc $0.\vec{x} = \vec{0}_E$
- $\lambda.\vec{0}_E = \lambda.(\vec{0}_E + \vec{0}_E) = \lambda.\vec{0}_E + \lambda.\vec{0}_E$, donc $\lambda.\vec{0}_E = \vec{0}_E$.
- $\Leftarrow$: ok.

  $\Rightarrow$. Supposons que $\lambda.\vec{x} = \vec{0}_E$.

  1^er cas : $\lambda = 0$.

  $\frac{1}{\lambda} \times \lambda = 1$

  2^nd cas : $\lambda \neq 0$.

  $\vec{x} = 1.\vec{x} = (\frac{1}{\lambda}\lambda).\vec{x} = \frac{1}{\lambda}.(\lambda.\vec{x}) = \frac{1}{\lambda}.\vec{0}_E = \vec{0}_E$.

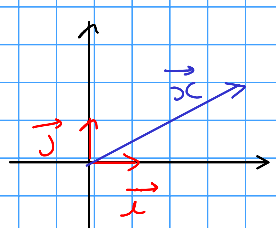
- $\vec{x} + (-1).\vec{x} = 1.\vec{x} + (-1).\vec{x}$

  $= \underbrace{(1+(-1))}_{=0}.\vec{x} = \vec{0}_E$.

  Si $x + y = e$, alors $y = -x$.

  Donc $\overrightarrow{-x} = (-1).\vec{x}$.

4. Combinaisons linéaires.



$\vec{x} = 3\vec{i} + 2\vec{j}$.

Donc $\vec{x}$ est combinaison linéaire de $\vec{i}$ et $\vec{j}$.

Soit $P \in \mathbb{K}_n[X]$.

$$P = a_0 1 + a_1 X + \dots + a_n X^n$$

$\lambda_0 \quad \lambda_1 \quad \lambda_n$

On cherche λ et μ tels que $(2,7) = \lambda(5,-2) + \mu(1,-3)$

lambda mu

$$= (5\lambda + \mu, -2\lambda - 3\mu).$$

$$\begin{cases} 5\lambda + \mu = 2 & \times 3 \\ -2\lambda - 3\mu = 7 \end{cases}$$

$13\lambda = 13$, $\lambda = 1$

$\mu = 2 - 5\lambda = -3$.

$$M = \lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ \lambda_1 & 0 \end{pmatrix} + \begin{pmatrix} \lambda_2 & -\lambda_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \lambda_3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 & -\lambda_2 \\ \lambda_1 & \lambda_3 \end{pmatrix}$$

$$\begin{cases} \lambda_1 + \lambda_2 = 2 \\ \lambda_1 = 1 \\ -\lambda_2 = 1 \\ \lambda_3 = 1 \end{cases} \quad \longrightarrow \lambda_1 = 1 \quad \longrightarrow \lambda_2 = -1$$

$\mathbb{K}[X]$: $(1, X, X^2, X^3, \dots, X^n, \dots)$

$(X^n)_{n \in \mathbb{N}}$

Soit $P \in \mathbb{K}[X]$. Alors $P = a_0 + a_1 X + \dots + a_n X^n$.

1.2. Sous-espace vectoriel.

1.2.1. Définition.

F est stable par + : $\forall(\vec{x},\vec{y}) \in F^2$, $\vec{x}+\vec{y} \in F$ $\qquad F+F \subset F$

F est stable par . : $\forall \vec{x} \in F \ \forall \lambda \in \mathbb{K} \quad \lambda.\vec{x} \in F$.

$. : \mathbb{K}\times E \to E \qquad ._{|F} : \mathbb{K}\times F \to F, \qquad +_{|F} : F\times F \to F.$

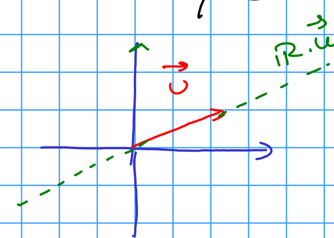
Remarque : un sous-espace vectoriel est un espace vectoriel.

Exemples 27.

- $\mathbb{R}.\vec{u} = \{\lambda.\vec{u}, \lambda \in \mathbb{R}\}$ sev de $\underbrace{\mathbb{R}^2}_{\text{est un e.v.}}$. $\qquad \vec{u}\begin{pmatrix} a \\ b \end{pmatrix}$

1. $\mathbb{R}.\vec{u} \subset \mathbb{R}^2$

   Soit $\vec{x} \in \mathbb{R}.\vec{u}$. Alors il existe $\lambda \in \mathbb{R}$ tel que
   $\vec{x} = \lambda.\vec{u} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix} \in \mathbb{R}^2$.
   Donc $\mathbb{R}.\vec{u} \subset \mathbb{R}^2$.



2. $\vec{0}_{\mathbb{R}^2} = (0,0) \in \mathbb{R}.\vec{u}$

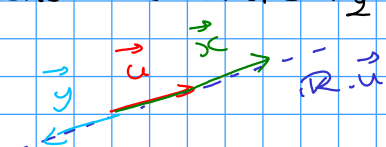
   $(0,0) = 0.\vec{u}$ avec $0 \in \mathbb{R}$

   Donc $\vec{0}_{\mathbb{R}^2} \in \mathbb{R}.\vec{u}$.

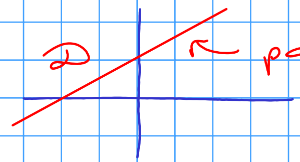
3. Soient $\vec{x},\vec{y} \in \mathbb{R}.\vec{u}$ et $\lambda \in \mathbb{R}$.

   (Montrons que $\lambda.\vec{x}+\vec{y} \in \mathbb{R}.\vec{u}$)

   On a $\vec{x} \in \mathbb{R}.\vec{u}$ donc il existe $\lambda_1 \in \mathbb{R}$ tel que $\vec{x} = \lambda_1.\vec{u}$. On a $\vec{y} \in \mathbb{R}.\vec{u}$ donc il existe $\lambda_2 \in \mathbb{R}$ tel que $\vec{y} = \lambda_2.\vec{u}$.



Donc
$$\begin{aligned}\lambda.\vec{x}+\vec{y} &= \lambda.(\lambda_1.\vec{u}) + \lambda_2.\vec{u} \\ &= (\lambda\lambda_1).\vec{u} + \lambda_2.\vec{u} \\ &= \underbrace{(\lambda\lambda_1+\lambda_2)}_{\in\mathbb{R}}.\vec{u} \in \mathbb{R}.\vec{u}.\end{aligned}$$



$\nwarrow$ pas un sev car $(0,0) \notin \mathcal{D}$

2. $\vec{0}_{\mathbb{R}^3} = (0,0,0) \in \mathcal{P}$? $\qquad \mathcal{P} = \{(\underset{0}{x}, \underset{0}{y}, \underset{0}{z}) \in \mathbb{R}^3,\ ax+by+cz=0\}$.

Si $(x,y,z) = (0,0,0)$ alors $ax+by+cz = a\times 0 + b\times 0 + c\times 0$
$= 0$.

Donc $\vec{0}_{\mathbb{R}^3} \in \mathcal{P}$.

3. Soient $\vec{x} = (x_1, x_2, x_3)$ et $\vec{y} = (y_1, y_2, y_3)$ éléments de $\mathcal{P}$
et $\lambda \in \mathbb{K}$.

Montrons que $\lambda\vec{x} + \vec{y} \in \mathcal{P}$.

$$= (\underbrace{\lambda x_1 + y_1}_{\text{“}x\text{”}}, \underbrace{\lambda x_2 + y_2}_{\text{“}y\text{”}}, \underbrace{\lambda x_3 + y_3}_{\text{“}z\text{”}})$$

On a $a(\lambda x_1 + y_1) + b(\lambda x_2 + y_2) + c(\lambda x_3 + y_3)$
$= a\lambda x_1 + ay_1 + b\lambda x_2 + by_2 + c\lambda x_3 + cy_3$
$= \lambda(\underbrace{ax_1 + bx_2 + cx_3}_{=0 \text{ car } \vec{x} \in \mathcal{P}})$
$+ \underbrace{ay_1 + by_2 + cy_3}_{=0 \text{ car } \vec{y} \in \mathcal{P}}$.

$= 0$.

On sait que
$ax_1 + bx_2 + cx_3 = 0$
car $\vec{x} \in \mathcal{P}$.

et $ay_1 + by_2 + cy_3 = 0$.

1. $F \subset \mathbb{R}^3$.

2. $\vec{0}_{\mathbb{R}^3} = (\underset{x}{0}, \underset{y}{0}, \underset{z}{0}) \in F$ car $xy = 0 \times 0 = 0$

3. $\vec{x} = (\underset{x}{1}, \underset{y}{0}, \underset{z}{0}) \in F$ car $xy = 1 \times 0 = 0$

$\vec{y} = (\underset{x}{0}, \underset{y}{1}, \underset{z}{0}) \in F$ car $xy = 0 \times 1 = 0$

Mais $\vec{x} + \vec{y} = (\underset{x}{1}, \underset{y}{1}, \underset{z}{0}) \notin F$ car $xy = 1 \times 1 = 1 \neq 0$

$\lambda_1, \dots \lambda_n \in \mathbb{K}$, $\vec{x}_1, \dots \vec{x}_n \in F$, $\underbrace{\lambda_1\vec{x}_1 + \lambda_2\vec{x}_2}_{\in F} + \underbrace{\lambda_3\vec{x}_3}_{\in F} \in F$ ...

$\lambda_1\vec{x}_1 + \dots + \lambda_n\vec{x}_n \in F$.