GEOM 2

Chap 2. Les polynômes

Cours 12 (1)

I. Arithmétique des polynômes

2. PGCD et PPCM

$A\mathbb{K}[X] + B\mathbb{K}[X]$ est un idéal de $\mathbb{K}[X]$

idéal idéal

unitaire.

$\underbrace{A\mathbb{K}[X] + B\mathbb{K}[X]}_{I = (D)} = D\,\mathbb{K}[X]$

$A\mathbb{K}[X] \cap B\mathbb{K}[X]$ est un idéal de $\mathbb{K}[X]$

$= (M) = M\mathbb{K}[X] = \{MP,\ P \in \mathbb{K}[X]\}$

unitaire.

$A = BQ + R$.

$\mathrm{pgcd}(A,B) = \mathrm{pgcd}(B,R)$.

$A\mathbb{K}[X] + B\mathbb{K}[X] = \mathrm{pgcd}(A,B)\mathbb{K}[X]$.

$R_0 \quad A = \overset{R_1}{B}Q + R^{R_2}$. $\qquad \deg R_2 < \deg B = \deg R_1$

$R_1 \quad B = R_2 Q_2 + R_3 \qquad \deg R_3 < \deg R_2$

$R_2 = R_3 Q_3 + R_4 \qquad \deg R_4 < \deg R_3$

$R_3 = R_4 Q_4 + R_5 \qquad \deg R_5 < \deg R_4$

$\vdots \qquad \vdots$

$R_{n-1} = R_n Q_n + 0$

Si $\deg A < \deg B$, $\quad A = 0 \times B + A$ et $\deg A < \deg B$.

$B = AQ + R$ ...

Ex 78. $A = X^4 + X^3 + 2X^2 + X + 1$ et $B = X^3 - 3X^2 + X - 3$.

$$\begin{array}{l|l} X^4 + X^3 + 2X^2 + X + 1 & X^3 - 3X^2 + X - 3 \\ \underline{X^4 - 3X^3 + X^2 - 3X} & \overline{X + 4} \\ \quad 4X^3 + X^2 + 4X + 1 & \\ \quad \underline{4X^3 - 12X^2 + 4X - 12} & \\ \qquad 13X^2 + 13 & \end{array}$$

Donc $A = (X+4)B + 13X^2 + 13$.

$$\begin{array}{l|l} X^3 - 3X^2 + X - 3 & 13X^2 + 13 \\ \underline{X^3 \quad 0 \quad + X} & \overline{\frac{1}{13}X - \frac{3}{13}} \\ \quad -3X^2 \quad -3 & \\ \quad \underline{-3X^2 \quad -3} & \\ \qquad 0 & \end{array}$$

Donc $A = (X+4)B + 13X^2 + 13$.

$B = (13X^2 + 13) \times \frac{1}{13}(X-3) + 0$.

Donc $\text{pgcd}(A, B) = X^2 + 1$.

2. Polynômes premiers entre eux

Cours 12 ②

Si $D \mid X - a$ et $D \mid X - b$ alors $D \mid b - a$

$1\mathbb{K}[X] = \{1 \times P,\ P \in \mathbb{K}[X]\} = \mathbb{K}[X]$.

$AC\mathbb{K}[X] = \{ACP,\ P \in \mathbb{K}[X]\} \subset \{AQ,\ Q \in \mathbb{K}[X]\} = A\mathbb{K}[X]$

$BC\mathbb{K}[X] = AD\mathbb{K}[X] \subset A\mathbb{K}[X]$.

$BC = AD$

Si α est racine de P de multiplicité m alors α est racine de P' de multiplicité $m-1$.

$X - \alpha \mid P'$ et $X - \alpha \mid P$. Donc $X - \alpha \mid \text{pgcd}(P, P')$

Donc $\mathrm{pgcd}(P,P') \neq 1$.

Chapitre 3 : Fractions rationnelles

Cours 12 ③

$P = \lambda \neq 0$, $Q = \dfrac{1}{\lambda}$ $\quad PQ = 1$. $\quad \dfrac{1}{P}$

$PQ = 1$. $\quad \underbrace{\deg P}_{=0} + \deg Q = 0$.

$\mathbb{Z}$: $-2, -1, 0, 1, 2 \ldots$ $\qquad \mathbb{Q}$ $\quad \dfrac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$.

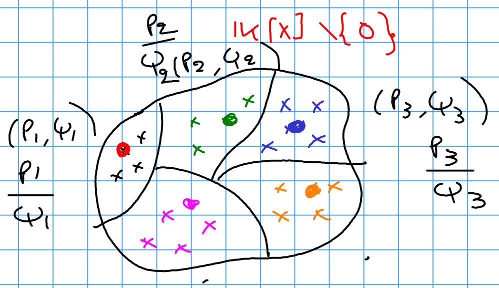
$\mathbb{K}[X]$ $\qquad \mathbb{K}(X)$ $\quad \dfrac{P}{Q}$, $P, Q \in \mathbb{K}[X]$, $Q \neq 0$.

$f(x) = \dfrac{2x}{x^2+1}$ $\qquad R = \dfrac{2X}{X^2+1}$.

1. Le corps des fractions rationnelles.

Sur $\mathbb{K}[X] \times \underbrace{\mathbb{K}[X]^*}_{\mathbb{K}[X] \setminus \{0\}}$, $\quad (P,Q) \sim (R,S)$

ssi $PS = QR$.



$\mathbb{K}(X) = \mathbb{K}[X] \times \mathbb{K}[X]^* / \sim$

$(\mathbb{K}(X), +, \times)$ anneau commutatif et tout élément de $\mathbb{K}(X)^*$ est inversible par $\times$.

$\dfrac{A}{B} + \underbrace{\dfrac{0}{1}}_{= 0 \in \mathbb{K}(X)} = \dfrac{A + 0 \times B}{B} = \dfrac{A}{B} = \dfrac{0}{1} + \dfrac{A}{B}$.

$\dfrac{A}{B} \times \dfrac{1}{\underbrace{1}_{1 \in \mathbb{K}[X]}} = \dfrac{A \times 1}{B \times 1} = \dfrac{A}{B} = \dfrac{1}{1} \times \dfrac{A}{B}$.

$$\frac{A}{B} \neq \frac{0}{1}.$$

$$\frac{A}{B} \times \frac{B}{A} = \frac{AB}{BA} = \frac{1}{1}.$$

$$\frac{A}{B} + \frac{-A}{B} = \frac{A-A}{B} = \frac{0}{B} = \frac{0}{1}.$$

$$F = \frac{X+1}{X-1} \quad \text{et} \quad G = \frac{X^2-2X-1}{X(X+1)}.$$

$$F+G = \frac{X+1}{X-1} + \frac{X^2-2X-1}{X(X+1)} = \frac{(X+1)^2 X + (X-1)(X^2-2X-1)}{(X-1)X(X+1)}$$

$$= \frac{(X^2+2X+1)X + X^3 - 2X^2 - X - X^2 + 2X + 1}{X(X-1)(X+1)}$$

$$= \frac{2X^3 - X^2 + 2X + 1}{X(X-1)(X+1)}.$$

$$F \times G = \frac{X+1}{X-1} \times \frac{X^2-2X-1}{X(X+1)} = \frac{(X+1)(X^2-2X-1)}{(X-1)X(X+1)}.$$

$$2 \in \mathbb{Z} \subset \mathbb{Q} \qquad\qquad P \in \mathbb{K}[X] \subset \mathbb{K}(X)$$

$$= \frac{2}{1} \qquad\qquad = \frac{P}{1}.$$

Si $f(P) = f(Q)$ alors $\frac{P}{1} = \frac{Q}{1}$ donc $P \times 1 = Q \times 1$,

$P = Q$.

$$\frac{15}{9} = \frac{3 \times 5}{3 \times 3} = \frac{5}{3}$$ pgcd$(5,3)$.

Ex 9. $\text{Pgcd}\left(2X(X+1)^2, (X+1)(X+2)\right) = X+1 \neq 1.$

$$R = \frac{2X(X+1)^2}{(X+1)(X+2)} = \frac{2X(X+1)}{X+2}.$$

$$\left(\frac{1}{(X-a)^n}\right)' = -\frac{n(X-a)^{n-1}}{(X-a)^{\underbrace{2n}_{n-1+n+1}}} = -\frac{n}{(X-a)^{n+1}}.$$

$$R = \frac{A}{B} \quad , \quad \frac{1}{R} = \frac{B}{A}$$

$$\deg R = \deg A - \deg B = -(\deg B - \deg A) = -\deg\left(\frac{1}{R}\right).$$

$$X^2 - 3 = (X - \sqrt{3})(X + \sqrt{3}).$$

$$R = \frac{X+1}{X}, \quad R' = \frac{X - (X+1)}{X^2} = \frac{-1}{X^2} \quad , \quad \deg R' = -2$$

$\deg 0$.