GEOM 2

Chapitre 3 : Fractions rationnelles

Cours 13 ①

$R(\lambda) = 0$

$A(\lambda) = 0$ $\qquad R(\lambda) = \dfrac{A(\lambda)}{B(\lambda)} = 0$

$B(\lambda) \neq 0$ car A et B sont premiers entre eux

$R = \dfrac{(X-1)^4}{X+2}$.

$R = \dfrac{(X-\lambda)\tilde{A}}{(X-\lambda)\tilde{B}}$

$B(\mu) = 0$. $\qquad R = \dfrac{3X}{(X-1)(X-2)^2}$

$X^2 + X + 1 = (X-j)(X-\bar{j})$ où $j = e^{2i\pi/3}$

$\bar{j} = e^{-\frac{2i\pi}{3}} = e^{\frac{4i\pi}{3}} = j^2$.

$R = \dfrac{(X+1)^2(X-1)^2 X}{(X-1)(X-2)(X+3)} \in \mathbb{R}(X)$.

$\mathbb{R}\setminus\{1,2,-3\} \longrightarrow \mathbb{R}$.

$x \longmapsto \dfrac{(x+1)^2(x-1)^2 x}{(x-1)(x-2)(x+3)}$

Ex 24.

$R = \dfrac{X^3 + 2X^2 + X + 1}{X+1} = E + \underbrace{S}_{\deg S < 0}$

$S = \dfrac{P}{Q}$

$\deg S = \deg P - \deg Q$.

$\deg P < \deg Q$

$\deg R = 3 - 1 = 2 > 0$.

(Si $\deg R < 0$ alors $E = 0$ et $S = R$).

$$\begin{array}{rl|l}
X^3 + 2X^2 + X + 1 & & X+1 \\
X^3 + X^2 & & \overline{X^2 + X} \\
\hline
X^2 + X + 1 & & \\
X^2 + X & & \\
\hline
1 & &
\end{array}$$

Donc $X^3 + 2X^2 + X + 1 = (X+1)(X^2+X) + 1$.

Donc $R = \dfrac{(X+1)(X^2+X) + 1}{X+1}$

$$R = \frac{A}{B}$$

$$= \underbrace{X^2 + X}_{E \in \mathbb{K}[X]} + \underbrace{\frac{1}{X+1}}_{\deg S < 0} \quad S.$$

2_ Décompositions en éléments simples

Cours 13 ②

polynômes de degré 0.

$$\frac{X+2}{(X+1)(X-2)} = \frac{a}{X+1} + \frac{b}{X+2}$$

$$\frac{R_1}{Q_1} - \frac{S_1}{Q_1} = \frac{S_2}{Q_2} - \frac{R_2}{Q_2}$$

$$\frac{R_1 - S_1}{Q_1} = \frac{S_2 - R_2}{Q_2}, \quad Q_2(R_1 - S_1) = Q_1(S_2 - R_2)$$

$$Q_1 \mid Q_2(R_1 - S_1), \quad Q_1 \mid R_1 - S_1 \quad R_1 - S_1 = AQ_1$$

$$\deg(R_1 - S_1) \leq \max(\underbrace{\deg(R_1)}_{< \deg Q_1}, \underbrace{\deg S_1}_{< \deg Q_1}) < \deg Q_1 .$$

Donc $R_1 = S_1$.

$$F = \frac{X \; P}{\underbrace{(X-1)}_{Q_1}\underbrace{(X-2)}_{Q_2}} = \frac{R_1}{X-1} + \frac{R_2}{X-2} \quad \begin{aligned} &\deg R_1 < \deg X - 1 \\ &= 1 \end{aligned}$$

$$= \frac{a}{X-1} + \frac{b}{X-2}, \quad R_1 = a, \; R_2 = b$$

$$(X-1)\frac{X}{(X-1)(X-2)} = (X-1)\frac{a}{X-1} + \frac{b}{X-2}(X-1).$$

$$\frac{X}{X-2} = a + \frac{b(X-1)}{X-2}.$$

$X = 1 \qquad -1 = a + 0, \qquad a = -1.$

$$\frac{(X-2)\,X}{(X-1)(X-2)} = \frac{(X-2)\,a}{X-1} + \frac{b\,(X-2)}{X-2}.$$

$$0 = -a - \frac{b}{2}$$

$$b = -2a$$
$$= 2.$$

$$\frac{X}{X-1} = \frac{(X-2)a}{X-1} + b.$$

$$\frac{2}{1} = b, \quad b = 2.$$

$$\frac{X-1}{X(X+1)(X+5)} = \frac{a}{X} + \frac{b}{X+1} + \frac{c}{X+5}$$

$$\frac{\overset{P}{X+1}}{\underset{\varphi}{(X-3)}^{4=n}} = \frac{a_4}{(X-3)^4} + \frac{a_3}{(X-3)^3} + \frac{a_2}{(X-3)^2} + \frac{a_1}{X-3}$$

avec $a_i \in \mathbb{R}$.

$$\deg A_i < \deg(X-3) = 1.$$

Ex 29 $\quad F = \dfrac{2X+1}{(X-1)^2}$.

$P = 2X+1$,

$\varphi = X-1$, $n = 2$

$$F = \frac{A_2}{\varphi^2} + \frac{A_1}{\varphi} \quad \text{et} \quad \deg A_2 < \deg \varphi = 1$$
$$\deg A_1 < \deg \varphi = 1$$

$$= \frac{a_2}{(X-1)^2} + \frac{a_1}{X-1} \quad \text{avec } a_1, a_2 \in \mathbb{R}.$$

$$\frac{2X+1}{(X-1)^2} = \frac{3}{(X-1)^2} + \frac{a_1}{X-1}.$$

$$2X+1 = a_2 + a_1(X-1)$$

$$1 = a_2 - a_1$$

$$a_1 = a_2 - 1$$
$$= 2.$$

$X = 1$, $\quad 3 = a_2 + 0$, $\quad$ donc $\quad a_2 = 3$.

2. Décompositions en éléments simples

Méthode

(1) Partie entière : - Si $\deg R < 0$ alors $E = 0$

- Sinon, division euclidienne de P par Q.

$$R = E + \frac{\tilde{P}}{Q} \quad \text{et} \quad \deg \frac{\tilde{P}}{Q} < 0.$$

(2) $Q = P_1^{\alpha_1} \cdots P_r^{\alpha_r}$ avec $\alpha_i \in \mathbb{N}^*$, P_i polynômes irr. 2 à 2 distincts.

(3)
$$R = E + \frac{\tilde{P}}{P_1^{\alpha_1} \cdots P_r^{\alpha_r}}$$
$$= E + \frac{\tilde{P}_1}{P_1^{\alpha_1}} + \cdots + \frac{\tilde{P}_r}{P_r^{\alpha_r}} \quad \text{(cor 27)}$$
(prop 28)
$$= E + \frac{A_1}{P_1^{\alpha_1}} + \frac{A_2}{P_1^{\alpha_1 - 1}} + \cdots + \frac{A_{\alpha_1 - 1}}{P_1} \qquad \deg A_i < \deg P_1$$
$$+ \cdots$$
$$+ \frac{R_1}{P_r^{\alpha_r}} + \frac{R_2}{P_r^{\alpha_r - 1}} + \cdots + \frac{R_{\alpha_r - 1}}{P_r}. \qquad \deg R_i < \deg P_r.$$

$$F = \frac{X^2 + 3X + 1}{(X-1)^2 (X-2)}$$

(1) $\deg F = 2 - 3 = -1 < 0$ donc $E = 0$

(2) $Q = \underbrace{(X-1)^2}_{Q_1^2} \underbrace{(X-2)}_{Q_2}$ est décomposé en produit d'irréductibles sur $\mathbb{R}$.

(3)
$$F = \frac{A_1}{Q_1^2} + \frac{A_2}{Q_2}$$
$$= \frac{B_2}{Q_1^2} + \frac{B_1}{Q_1} + \frac{A_2}{Q_2}.$$

$\deg B_2 < 1$

$\deg B_1 < 1$

$\deg A_2 < 1$

$$\frac{X^2+3X+1}{(X-1)^2(X-2)} = \frac{a}{(X-1)^2} + \frac{b}{X-1} + \frac{c}{X-2}, \qquad a, b, c \in \mathbb{R}.$$

- $\dfrac{1^2+3\times 1+1}{1-2} = a$, donc $a = -5$.

- $\dfrac{4+6+1}{1} = 0 + c$, donc $c = 11$.

- $1 = 0 + b + c$, donc $b = 1 - c = -10$.

<u>ou</u> $-\dfrac{1}{2} = a - b - \dfrac{c}{2}$, donc $b = a - \dfrac{c}{2} + \dfrac{1}{2}$

$$= -5 - \frac{11}{2} + \frac{1}{2} = -10.$$

Donc $F = \dfrac{-5}{(X-1)^2} - \dfrac{10}{X-1} + \dfrac{11}{X-2}$.