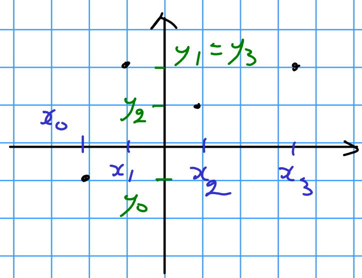
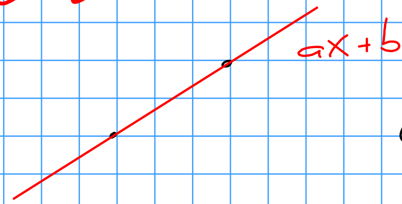
3. Polynômes d'interpolation de Lagrange





$(x_0, y_0), (x_1, y_1), (x_2, y_2),$
(x_3, y_3).

$\forall i \in \{0..3\} \quad P(x_i) = y_i \quad$ et $P \in \mathbb{K}_3[X]$

$y_0 = 1$
$y_1 = y_2 = y_3 = 0$

$y_i = 1$
$\forall j \neq i \quad y_j = 0$.

$P(x_i) = 1 \qquad$ et $P \in \mathbb{K}_n[X]$
$\forall j \neq i \quad P(x_j) = 0$

P est de degré $\leq n$ et $\forall j \in \{0..n\} \setminus \{i\} \quad P(x_j) = 0$

Donc P admet au moins n racines $\underbrace{\ }_{\text{distinctes}}$: $x_0, .. x_{i-1}, x_{i+1}, .. x_n$.

$P \neq 0$ car $P(x_i) = 1$.

Donc P est de degré n et admet n racines : $x_0, .. \cancel{x_i} .. x_n$.

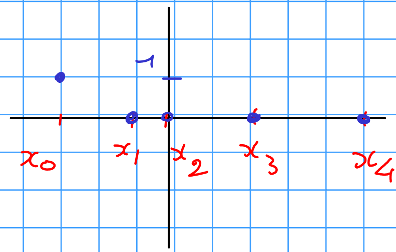
$$P = \lambda (X - x_0) .. (X - x_{i-1})(X - x_{i+1}) .. (X - x_n) \quad \text{avec } \lambda \in \mathbb{K}^*.$$

Donc $1 = P(x_i) = \lambda (x_i - x_0)(x_i - x_{i-1})(x_i - x_{i+1}) .. (x_i - x_n)$.

Donc $\lambda = \dfrac{1}{\prod\limits_{\substack{k=0 \\ k \neq i}}^{n} (x_i - x_k)}$ (bien défini car $x_i \neq x_k$ pour tout $k \in \{0..n\} \setminus \{i\}$.

Donc $P = \dfrac{1}{\prod\limits_{\substack{k=0..n \\ k \neq i}} (x_i - x_k)} \underbrace{(X - x_0) .. (X - x_{i-1})(X - x_{i+1}) .. (X - x_n)}_{\prod\limits_{\substack{k=0 \\ k \neq i}}^{n} (X - x_k)}$

$$= \prod_{\substack{k=0..n \\ k \neq i}} \frac{(X - x_k)}{(x_i - x_k)}$$



$L_0 \in \mathbb{R}_4[X]$.

$$L_i(x_0) = \begin{cases} 1 & \text{si } i = 0 \\ 0 & \text{sinon} \end{cases}$$

$$\lambda_0 L_0 + \lambda_1 L_1 + \cdots + \lambda_n L_n = 0$$

$$\lambda_0 \underbrace{L_0(x_0)}_{1} + \lambda_1 \underbrace{L_1(x_0)}_{0} + \cdots + \lambda_n \underbrace{L_n(x_0)}_{0} = 0$$

$$\lambda_0 = 0.$$

$$P = \lambda_0 L_0 + \lambda_1 L_1 + \cdots + \lambda_n L_n$$

$$P(x_0) = \lambda_0 \underbrace{L_0(x_0)}_{1} + \lambda_1 \underbrace{L_1(x_0)}_{0} + \cdots + \lambda_n \underbrace{L_n(x_0)}_{0}$$

Donc $P(x_0) = \lambda_0$, $P(x_1) = \lambda_1$, ...

$$\lambda_i = P(x_i).$$

$$P = P(x_0)L_0 + P(x_1)L_1 + \cdots + P(x_n)L_n.$$

$P(x_i) = y_i$.

$$P = y_0 L_0 + y_1 L_1 + \cdots + y_n L_n.$$

$$P(x_0) = y_0 \underbrace{L_0(x_0)}_{=1} + y_1 \underbrace{L_1(x_0)}_{=0} + \cdots + y_n \underbrace{L_n(x_0)}_{=0}$$

$$= y_0.$$

$$P(x_i) = y_i.$$

$P(x_i) = y_i = Q(x_i)$ et $P, Q \in \mathbb{K}_n[X]$.

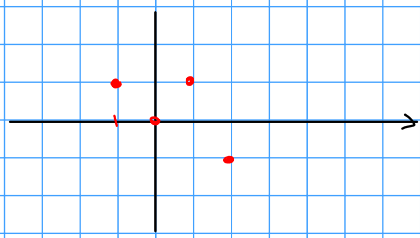
$(P-Q)(x_i) = P(x_i) - Q(x_i) = 0 \quad \forall i \in \{0 \ldots n\}$

Donc $P - Q$ admet $n+1$ racines et $\deg(P-Q) \leq n$

Donc $P - Q = 0$, donc $P = Q$.

$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 2$

$y_0 = 1, \quad y_1 = 0, \quad y_2 = 1, \quad y_3 = -1.$



$L(x_i) = y_i$.

$$L = \sum_{i=0}^{3} y_i L_i = 1 \times L_0 + 0 \times L_1 + 1 \times L_2 + (-1) L_3$$

$$= L_0 + L_2 - L_3 .$$

$$L_i = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x - x_k)}{(x_i - x_k)} .$$

$$L_0 = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} = \ldots$$

$$L_2 = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} = \frac{(x-(-1))(x-0)(x-2)}{(1-(-1))(1-0)(1-2)} = \ldots$$

$f(x)$

4. Polynômes irréductibles

1. Définition et décomposition

$$D = a_1X + b_1 \qquad a_1 \neq 0$$

$$= \underbrace{a_1}_{\lambda_1}\left(X + \underbrace{\frac{b_1}{a_1}}_{-\alpha_1}\right)$$

$$P = DQ \quad , \quad \underbrace{\deg P}_{=2 \text{ ou } 3} = \underbrace{\deg D}_{\geqslant 1} + \underbrace{\deg Q}_{\geqslant 1}$$

$\deg D = 1$ ou $\deg Q = 1$.

Si $\deg D = 1$, $D = aX + b = a\left(X + \frac{b}{a}\right)$

donc $-\frac{b}{a}$ est racine de D donc de P.

$$\left(\frac{p}{q}\right)^2 - 2 = 0 \qquad p^2 = 2q^2 \ . \quad \dots \quad \text{Absurde} \ .$$

$$P = a_nX^n + a_{n-1}X^{n-1} + \dots + a_0$$

$$= \underset{\lambda}{a_n}\underbrace{\left(X^n + \frac{a_{n-1}}{a_n}X^{n-1} + \dots + \frac{a_0}{a_n}\right)}_{P_1 \text{ irr et unitaire.}}$$

$$n+1 = \deg P = \underbrace{\deg A}_{\geqslant 1} + \underbrace{\deg B}_{\geqslant 1}$$

1 est racine de $X^3 - 1$.

Donc $X - 1$ divise $X^3 - 1$. Puis par division euclidienne, on trouve $X^3 - 1 = (X-1)(X^2 + X + 1)$.

2. Polynômes irréductibles sur $\mathbb{C}$ et $\mathbb{R}$.

$$P = \lambda \prod_{i=1}^{r}(X - \alpha_i)^{m_i} \qquad \alpha_i \notin \mathbb{R} \ .$$

Si $\alpha \in \mathbb{C}$ et $P \in \mathbb{R}[X]$ alors $\overline{\alpha}$ est aussi racine de P.
et α racine de P

$$P = \lambda \prod_{i=1}^{s} \Big(\underbrace{X - \overset{\notin \mathbb{R}}{\beta_i}}_{\notin \mathbb{R}[X]}\Big)^{n_i} \Big(\underbrace{X - \overline{\beta_i}}_{\notin \mathbb{R}[X]}\Big)^{n_i} = \lambda \prod_{i=1}^{s} \Big((X-\beta_i)(X-\overline{\beta_i})\Big)^{n_i}$$

$$(X-\beta_i)(X-\overline{\beta_i}) = X^2 - (\beta_i + \overline{\beta_i})X + \beta_i \overline{\beta_i}$$
$$= X^2 - 2\,\mathrm{Re}(\beta_i)X + |\beta_i|^2 \in \mathbb{R}[X].$$

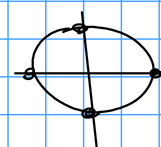
$$z^4 + 1 = 0$$

ssi $z^4 = -1 = e^{i\pi}$

ssi $\left(\dfrac{z}{e^{i\pi/4}}\right)^4 = 1.$

ssi $\dfrac{z}{e^{i\pi/4}} \in \left\{ e^{\frac{2i\pi}{4}\times 0}, e^{\frac{2i\pi}{4}\times 1}, e^{\frac{2i\pi}{4}\times 2}, e^{\frac{2i\pi}{4}\times 3} \right\}$

$$= \left\{ 1, e^{\frac{i\pi}{2}}, e^{+i\pi}, e^{-\frac{i\pi}{2}} \right\}$$



ssi $z \in \left\{ e^{i\pi/4}, e^{\frac{3i\pi}{4}}, e^{-\frac{3i\pi}{4}}, e^{-\frac{i\pi}{4}} \right\}.$

$$X^2 - \underbrace{\left(e^{\frac{i\pi}{4}} + e^{-\frac{i\pi}{4}}\right)}_{2\cos\left(\frac{\pi}{4}\right)} X + \underbrace{e^{\frac{i\pi}{4}} \times e^{-\frac{i\pi}{4}}}_{1}$$

$\mathrm{Re}(e^{ix}) = \cos(x).$