TD13 - Algèbre 2

$p \circ p = p$

$p(p(x)) = p(x)$

Exercice 1.

- Si $x + p(x) = b$ alors $p(x+p(x)) = p(b)$

donc $p(x) + \underbrace{p(p(x))}_{p(x)} = p(b)$

donc $p(x) = \frac{p(b)}{2}$.

Donc $x = b - p(x) = b - \frac{p(b)}{2}$.

- Posons $x = b - \frac{p(b)}{2}$,

Alors $x + p(x) = b - \frac{p(b)}{2} + p(b) - \frac{\overbrace{p(p(b))}^{p(b)}}{2} = b$.

Exercice 2.

1. - $F \cap G = \{(0,0,0,0)\}$:

Soit $(x,y,z,t) \in F \cap G$.

On a $x = -y$ et $z = -t$.

On a $(x,y,z,t) = \lambda_1 (1,1,0,0) + \lambda_2 (1,1,1,1)$

$= (\lambda_1 + \lambda_2, \lambda_1 + \lambda_2, \lambda_2, \lambda_2)$ où $\lambda_1, \lambda_2 \in \mathbb{R}$.

Donc $\begin{cases} \lambda_1 + \lambda_2 = -\lambda_1 - \lambda_2 \\ \lambda_2 = -\lambda_2 \end{cases}$ donc $\begin{cases} \lambda_2 = 0 \\ \lambda_1 = 0 \end{cases}$.

Donc $(x,y,z,t) = (0,0,0,0)$.

- $\dim G = 2$

$F = \{(x,-x,z,-z),\ (x,z) \in \mathbb{R}^2\}$

$= \mathrm{Vect}((1,-1,0,0),(0,0,1,-1))$, donc $\dim F = 2$.

Donc $\dim F + \dim G = 4 = \dim \mathbb{R}^4$.

Donc $\mathbb{R}^4 = F \oplus G$.

2. Pour tout $(x,y,z,t) \in \mathbb{R}^4$,

$$(x,y,z,t) = \underbrace{(x_f, y_f, z_f, t_f)}_{\in F} + \underbrace{(x_g, y_g, z_g, t_g)}_{\in G}.$$

Alors $p((x,y,z,t)) = (x_f, y_f, z_f, t_f) = (x_f, -x_f, z_f, -z_f)$

Donc $(x,y,z,t) = (x_f, -x_f, z_f, -z_f) + \lambda_1(1,1,0,0) + \lambda_2(1,1,1,1)$

$$= (x_f + \lambda_1 + \lambda_2,\ -x_f + \lambda_1 + \lambda_2,\ z_f + \lambda_2,\ -z_f + \lambda_2).$$

$$\begin{cases} x = x_f + \lambda_1 + \lambda_2 \\ y = -x_f + \lambda_1 + \lambda_2 \\ z = z_f + \lambda_2 \\ t = -z_f + \lambda_2 \end{cases}$$

$$\lambda_2 = \frac{z+t}{2}$$

$$z_f = z - \frac{z+t}{2} = \frac{z-t}{2}.$$

$$\lambda_1 + \lambda_2 = \frac{x+y}{2}$$

$$\lambda_1 = \frac{x+y}{2} - \frac{z+t}{2}.$$

$$x_f = x - \frac{x+y}{2} = \frac{x-y}{2}.$$

Donc $p((x,y,z,t)) = \left(\frac{x-y}{2}, \frac{y-x}{2}, \frac{z-t}{2}, \frac{t-z}{2}\right)$.

Donc
$$\operatorname{mat}_{\mathcal{B}_c} p = \begin{pmatrix} \frac{1}{2} & -1/2 & 0 & 0 \\ -\frac{1}{2} & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{pmatrix}.$$

$$p((1,0,0,0)) = \left(\frac{1}{2}, -\frac{1}{2}, 0, 0\right),$$

$\vdots$

ou $\mathcal{B}_1 = \left(\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right)$ base de F

$\mathcal{B}_2 = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right)$ base de G.

$$\vec{x} = \underbrace{\vec{f}}_{\in F} + \underbrace{\vec{g}}_{\in G}$$

$$p(\vec{x}) = \vec{f}$$

Posons $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2) = \left(\underbrace{\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}}_{\in F}, \underbrace{\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}}_{\in F}, \underbrace{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}_{\in G}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \right) = (u_1, u_2, u_3, u_4)$

$$A = \operatorname{mat}_{\mathcal{B}} p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$p\left(\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\-1\\0\\0\end{pmatrix} = 1u_1 + 0u_2 + 0u_3 + 0u_4.$$

$u_2 = u_2 + 0$ $\quad p(u_2) = u_2.$

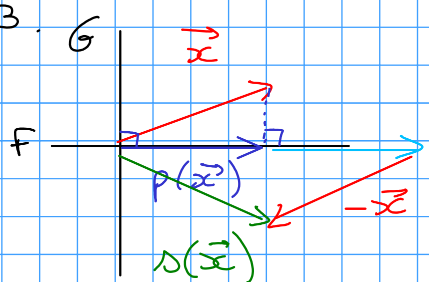
$u_3 = \underbrace{0}_{\in F} + \underbrace{u_3}_{\in G}$, $\quad p(u_3) = 0$

$p(u_2) = u_2$

$$\operatorname{mat}_{\mathcal{B}_c} p = \underbrace{P_{\mathcal{B}_c}^{\mathcal{B}}}_{P} \, A \, P_{\mathcal{B}}^{\mathcal{B}_c} = PAP^{-1}$$

$$P = \begin{pmatrix} 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$ Calculer P^{-1}, puis PAP^{-1}.

3.6



$s(\vec{x}) = 2p(\vec{x}) - \vec{x}$

$s = 2p - id$

$$\operatorname{mat}_{\mathcal{B}_c} s = 2 \operatorname{mat}_{\mathcal{B}_c} p - \operatorname{mat}_{\mathcal{B}_c} id$$

$$= 2\begin{pmatrix} \frac{1}{2} & -1/2 & 0 & 0 \\ -\frac{1}{2} & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$\vec{\imath}\ (1,0,0)$
$\vec{\jmath}\ (0,1,0)$
$\vec{k}\ (0,0,1)$
$\vec{u}\ (1,1,1)$

Exercice 4

1. $f((x,y,z)) = \left(\frac{2}{3}x - \frac{1}{3}y - \frac{1}{3}z,\ -\frac{1}{3}x + \frac{2}{3}y - \frac{1}{3}z,\ -\frac{1}{3}x - \frac{1}{3}y + \frac{2}{3}z\right)$.

$$A = \mathrm{mat}_{\mathcal{B}_c} f = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

2. Soit $(x,y,z) \in \mathbb{R}^3$.

$(x,y,z) \in \ker f$ ssi $\frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

ssi $\begin{cases} 2x - y - z = 0 \\ -x + 2y - z = 0 \\ -x - y + 2z = 0 \end{cases}$

ssi $\begin{cases} x + y - 2z = 0 \\ -x + 2y - z = 0 \\ \cancel{-x - y + 2z = 0} \end{cases}$

ssi $\begin{cases} x + y - 2z = 0 \\ y = z \end{cases}$

ssi $\begin{cases} y = z \\ x = y \end{cases}$

ssi $(x,y,z) = x\,(1,1,1)$.

Donc $\ker f = \mathrm{Vect}((1,1,1))$. $((1,1,1))$ base de $\ker f$.

On a $\dim \ker f = 1$ et $\dim \operatorname{Im} f = 3 - 1 = 2$.

$\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \in \operatorname{Im} f$, non colinéaires donc forment une famille libre de $\operatorname{Im} f$, composée de 2 éléments et $\dim \operatorname{Im} f = 2$.

Donc $((2,-1,-1),(-1,2,-1))$ base de $\operatorname{Im} f$.

- $\dim \ker f + \dim \operatorname{Im} f = 3$.

- $\ker f \cap \operatorname{Im} f = \{(0,0,0)\}$.

Soit $(x,y,z) \in \ker f \cap \operatorname{Im} f$.

$$(x,y,z) = \lambda_1(2,-1,-1) + \lambda_2(-1,2,-1)$$
$$= (2\lambda_1 - \lambda_2, -\lambda_1 + 2\lambda_2, -\lambda_1 - \lambda_2).$$

et $\begin{cases} 2\lambda_1 - \lambda_2 = -\lambda_1 - \lambda_2, \\ -\lambda_1 + 2\lambda_2 = -\lambda_1 - \lambda_2, \end{cases}$ donc $\begin{cases} \lambda_1 = 0, \\ \lambda_2 = 0. \end{cases}$

D'où $\ker f \oplus \operatorname{Im} f = \mathbb{R}^3$.

3. $V = \{(x,y,z) \in \mathbb{R}^3 \,/\, f((x,y,z)) = (x,y,z)\}$.

$= \ker g$ où $g : (x,y,z) \mapsto f((x,y,z)) - (x,y,z)$.

g est linéaire.

Donc V est un sev de $\mathbb{R}^3$.

Rem.
$F = \{M \in \mathcal{M}_n(\mathbb{R}), \operatorname{Tr} M = 0\}$
$F = \ker \operatorname{Tr}$ donc F est un sev.

$(x,y,z) \in V$

ssi $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

ssi $\begin{cases} 2x - y - z = x \\ -x + 2y - z = y \\ -x - y + 2z = z \end{cases}$ ssi $z = -x - y$.

Donc $V = \text{Vect}\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right)$.

Donc $\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right)$ base de V.

4. On a $\ker f \oplus \text{Im} f = \mathbb{R}^3$.

$\text{Im} f = \text{Vect}\left(\begin{pmatrix}2\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\-1\end{pmatrix}\right)$.

$\begin{pmatrix}2\\-1\\-1\end{pmatrix} \in V$ car $\begin{pmatrix}2\\-1\\-1\end{pmatrix} = 2\begin{pmatrix}1\\0\\-1\end{pmatrix} - \begin{pmatrix}0\\1\\-1\end{pmatrix}$

$\begin{pmatrix}-1\\2\\-1\end{pmatrix} \in V$ car $\begin{pmatrix}-1\\2\\-1\end{pmatrix} = 2\begin{pmatrix}0\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix}$.

Donc $f((2,-1,-1)) = (2,-1,-1)$,
$f((-1,2,-1)) = (-1,2,-1)$.

Donc pour tout $(x,y,z) \in \text{Im} f$, $f(x,y,z) = (x,y,z)$.

Donc pour tout $(x,y,z) \in \mathbb{R}^4 = \ker f \oplus \text{Im} f$,

$$(x,y,z) = \underbrace{(x_1,y_1,z_1)}_{\ker f} + \underbrace{(x_2,y_2,z_2)}_{\in \text{Im} f}$$

on a $f((x,y,z)) = \underbrace{f((x_1,y_1,z_1))}_{=0} + \underbrace{f((x_2,y_2,z_2))}_{(x_2,y_2,z_2)}$

$= (x_2,y_2,z_2)$.

Donc f est la projection sur $\text{Im} f$ // à $\ker f$.

5. $\mathcal{B}_1 = ((1,1,1))$ base de $\ker f$

$\mathcal{B}_2 = ((2,-1,-1),(-1,2,-1))$ base de $\text{Im} f$

$\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ base de $\mathbb{R}^3$ $\qquad \mathcal{B} = (\underbrace{u_1}_{\in \ker f}, \overset{\in \text{Im} f}{u_2}, \overset{\in \text{Im} f}{u_3})$

$$\text{mat}_{\mathcal{B}} f = \begin{array}{c} \begin{array}{ccc} f(u_1) & f(u_2) & f(u_3) \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow u_1 \\ \leftarrow u_2 \\ \leftarrow u_3 \end{array}$$

$f(u_2) = u_2$
$f(u_3) = u_3$

$$A = \text{mat}_{\mathcal{B}_c} f = \underbrace{P_{\mathcal{B}_c}^{\mathcal{B}}}_{P} \, \text{mat}_{\mathcal{B}} f \; P_{\mathcal{B}}^{\mathcal{B}_c}$$

Donc $A = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}$, P inversible car matrice de passage

d'où $P^{-1} A P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

avec $P = P_{\mathcal{B}_c}^{\mathcal{B}} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & -1 \end{pmatrix}$.