2. Décompositions en éléments simples

$$R = \frac{A}{B}$$

(1) Calculer E $\quad R = E + \frac{\tilde{A}}{B}$

(2) $B = \lambda (X-\lambda_1)^{\alpha_1} \cdots (X-\lambda_r)^{\alpha_r} (X^2+b_1X+c_1)^{\beta_1} \cdots (X^2+b_pX+c_p)^{\beta_p}$

(3)
$$\frac{\tilde{A}}{B} = \frac{a_1}{X-\lambda_1} + \frac{a_2}{(X-\lambda_1)^2} + \cdots + \frac{a_{\alpha_1}}{(X-\lambda_1)^{\alpha_1}} + \cdots + \frac{b_1}{X-\lambda_r} + \cdots + \frac{b_{\alpha_r}}{(X-\lambda_r)^{\alpha_r}} + \frac{c_1+d_1X}{X^2+b_1X+c} + \cdots + \frac{c_{\beta_1}+c_{\beta_1}X}{(X^2+b_1X+c_1)^{\beta_1}} + \cdots$$

(4) Déterminer les constantes aux numérateurs

- $G = \dfrac{4}{(X^2-1)^2}$. $\qquad X^2-1 = (X-1)(X+1)$

(1) $d^\circ 4 = 0 < d^\circ((X^2-1)^2) = 4$. Donc $E = 0$.

(2) $(X^2-1)^2 = ((X-1)(X+1))^2 = (X-1)^2(X+1)^2$

(3) Donc G s'écrit sous la forme

$$G = \frac{a}{X-1} + \frac{b}{(X-1)^2} + \frac{c}{X+1} + \frac{d}{(X+1)^2}$$

avec $(a,b,c,d) \in \mathbb{R}^4$.

(4) On a $G(-X) = G(X)$

$$G(-X) = \frac{-a}{X+1} + \frac{b}{(X+1)^2} + \frac{-c}{X-1} + \frac{d}{(X-1)^2}$$

$$= \frac{c}{X+1} + \frac{d}{(X+1)^2} + \frac{a}{X-1} + \frac{b}{(X-1)^2}$$

Donc $a = -c$ et $b = d$.

Donc $G = \dfrac{a}{X-1} + \dfrac{b}{(X-1)^2} + \dfrac{-a}{X+1} + \dfrac{b}{(X+1)^2}$.

$$\frac{4}{(X-1)^2(X+1)^2} = \frac{a}{X-1} + \frac{b}{(X-1)^2} + \frac{-a}{X+1} + \frac{b}{(X+1)^2}$$

Donc $1 = 0 + b + 0 + 0$, donc $b = 1$

En évaluant en 0, $4 = -a + b - a + b$,

Donc $2a = 2b - 4 = -2$, donc $a = -1$.

Donc $$G = \frac{-1}{X-1} + \frac{1}{(X-1)^2} + \frac{1}{X+1} + \frac{1}{(X+1)^2}$$

$R = \frac{A}{B}$

Pôle simple : λ est racine simple de B, $B = (X-\lambda)\tilde{B}$

$$R = \frac{A}{(X-\lambda)\tilde{B}}, \quad \tilde{B} = P_1^{\alpha_1} \dots P_r^{\alpha_r}$$

$$= \frac{A}{(X-\lambda)^1 P_1^{\alpha_1} \dots P_r^{\alpha_r}} = \frac{a}{X-\lambda} + \underbrace{\frac{A_1}{P_1} + \dots + \frac{A_{\alpha_1}}{P_1^{\alpha_1}} + \dots + \cdot}_{\tilde{R}}$$

$$a = \frac{A(\lambda)}{B'(\lambda)}$$

$$R = \frac{A}{B} = \frac{A}{(X-\lambda)C}. \quad \frac{A}{C} = (X-\lambda)R$$

$$= (X-\lambda)\left(\frac{a}{X-\lambda} + \varphi\right)$$

$$= a + (X-\lambda)\varphi.$$

Donc $\left(\frac{A}{C}\right)(\lambda) = \frac{A(\lambda)}{C(\lambda)} = a + (\lambda-\lambda)\varphi(\lambda) = a$.

$B = (X-\lambda)C$, donc $B' = C + (X-\lambda)C'$

donc $B'(\lambda) = C(\lambda) + (\lambda-\lambda)C'(\lambda)$

$= C(\lambda)$.

$$H = \frac{X^4+1}{X^3-1}.$$

① $\deg(X^4+1) = 4 \geq \deg(X^3-1) = 3$

$$\begin{array}{r|l} X^4 + 1 & X^3 - 1 \\ \underline{X^4 - X} & \overline{X} \\ X + 1 & \end{array} \quad , \quad X^4+1 = X(X^3-1) + X+1.$$

Donc $H = \dfrac{X(X^3-1) + X - 1}{X^3-1} = X + \dfrac{X+1}{X^3-1}$.

$\underline{\underline{\text{Sur } \mathbb{C}}}$ ② $X^3 - 1 = (X-1)(X^2+X+1) = (X-1)(X-j)(X-\bar{j})$

$$\frac{\overset{A}{X+1}}{\underset{B}{X^3-1}} = \frac{a}{X-1} + \frac{b}{X-j} + \frac{c}{X-\bar{j}}, \quad \text{où } a, b, c \in \mathbb{R}.$$

$$\frac{X+1}{(X-1)(X-j)(X-\bar{j})}$$

$$\left(a = \frac{2}{(1-j)(1-\bar{j})} = \cdots \qquad j^3 = 1, \quad \frac{1}{j^2} = j, \quad 1+j+j^2 = 0 \right)$$

$$a = \frac{A(1)}{B'(1)} = \frac{2}{3}.$$

$$b = \frac{A(j)}{B'(j)} = \frac{1+j}{3j^2} = \frac{(1+j)j}{3} = \frac{j+j^2}{3} = \frac{-1}{3}.$$

$$c = \frac{A(\bar{j})}{B'(\bar{j})} = \frac{1+\bar{j}}{3\bar{j}^2} = \overline{\frac{1+j}{3j^2}} = \overline{-\frac{1}{3}} = -\frac{1}{3}.$$

Donc $\dfrac{X+1}{X^3-1} = \dfrac{1}{3}\left(\dfrac{2}{X-1} - \dfrac{1}{X-j} - \dfrac{1}{X-\bar{j}} \right)$.

$$\overline{\left(\frac{a}{b}\right)} = \frac{\bar{a}}{\bar{b}} \qquad \overline{a+b} = \bar{a} + \bar{b}$$

$$H = \frac{a}{X-1} + \frac{b}{X-j} + \frac{c}{X-\bar{j}}$$

$$H = \overline{H} = \frac{\bar{a}}{X-1} + \frac{\bar{b}}{X-\bar{j}} + \frac{\bar{c}}{X-j}$$

$$= \frac{\bar{a}}{X-1} + \frac{\bar{c}}{X-j} + \frac{\bar{b}}{X-\bar{j}}$$

$a \in \mathbb{R}$

$b = \bar{c}$

Donc $H = X + \dfrac{1}{3}\left(\dfrac{2}{X-1} - \dfrac{1}{X-j} - \dfrac{1}{X-\bar{j}} \right)$ dans $\mathbb{C}(X)$.

Puis, dans $\mathbb{R}(X)$, $H = X + \frac{1}{3}\left(\frac{2}{X-1} - \frac{X-\bar{j}+X-j}{(X-j)(X-\bar{j})}\right)$

$$= X + \frac{1}{3}\left(\frac{2}{X-1} - \frac{2X+1}{X^2+X+1}\right)$$

• $J = \dfrac{1}{(X-1)^2(X^2+4)}$.

$(X-1)^2(X^2+4)$

$= (X-1)^2(X-2i)(X+2i)$

$J = \dfrac{\tilde{a}}{X-1} + \dfrac{\tilde{b}}{(X-1)^2} + \dfrac{\tilde{c}}{X-2i} + \dfrac{\tilde{d}}{X+2i}$.

dans $\mathbb{C}(X)$.

$$\frac{1}{(X-1)^2(X^2+4)} = \frac{a}{X-1} + \frac{b}{(X-1)^2} + \frac{cX+d}{X^2+4}$$

On a $\frac{1}{5} = b$,

$$\frac{1}{(X-1)^2} = \frac{a}{X-1}(X^2+4) + \frac{b}{(X-1)^2}(X^2+4) + cX+d.$$

Pour $X = 2i$, $\dfrac{1}{(2i-1)^2} = 2ic+d$

$$\frac{1}{(2i-1)^2} = \frac{1}{-3-4i} = \frac{(-3+4i)}{25} = -\frac{3}{25} + \frac{4}{25}i = d + 2ci$$

$d = -\frac{3}{25}$, $c = \frac{2}{25}$.

$0 = a + c$, $a = -\frac{2}{25}$.

$$\left(2\begin{pmatrix}1&2\\0&1\end{pmatrix}\right)\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}1&2\\0&1\end{pmatrix}\times\left(2\times\begin{pmatrix}0&0\\1&0\end{pmatrix}\right)$$
$$=2\times\left(\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}\right).$$

$$\varphi(1)=\overline{1}=1$$
$$\varphi(xy)=\overline{xy}=\overline{x}\,\overline{y}=\varphi(x)\varphi(y)$$
$\lambda\in\mathbb{R}$, $\quad\varphi(\lambda x+y)=\overline{\lambda x+y}=\overline{\lambda}\,\overline{x}+\overline{y}=\lambda\,\overline{x}+\overline{y}=\lambda\varphi(x)+\varphi(y)$.

$$\varphi(id_E)=I_n$$
$$\varphi(u\circ v)=mat_{\mathcal{B}}(u\circ v)=mat_{\mathcal{B}}\,u\ mat_{\mathcal{B}}\,v=\varphi(u)\times\varphi(v)$$
$(\mathcal{L}(E),+,\circ,\cdot)$ $\qquad$ $(\mathcal{M}_n(\mathbb{K}),+,\times,\cdot)$

$$\varphi(\lambda u+v)=mat_{\mathcal{B}}(\lambda u+v)=\lambda\,mat_{\mathcal{B}}\,u+mat_{\mathcal{B}}\,v=\lambda\varphi(u)+\varphi(v).$$